

Chiral order and fluctuations in multi-flavour QCD

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Received: 31 July 2002 / Revised version: 6 November 2002 /

Published online: 15 January 2003 – © Springer-Verlag / Società Italiana di Fisica 2003

Abstract. Multi-flavour ($N_f \geq 3$) chiral perturbation theory (χ PT) may exhibit instabilities due to vacuum fluctuations of sea $\bar{q}q$ pairs. Keeping the fluctuations small would require a very precise fine tuning of the low-energy constants $L_4(\mu)$ and $L_6(\mu)$ to $L_4^{\text{crit}}(M_\rho) = -0.51 \cdot 10^{-3}$, $L_6^{\text{crit}}(M_\rho) = -0.26 \cdot 10^{-3}$. A small deviation from these critical values – like the one suggested by the phenomenology of OZI-rule violation in the scalar channel – is amplified by huge numerical factors inducing large effects of vacuum fluctuations. This would lead in particular to a strong N_f dependence of chiral symmetry breaking (χ SB) and a suppression of the multi-flavour chiral order parameters. A simple resummation is shown to cure the instability of $N_f \geq 3$ χ PT, but it modifies the standard expressions of some $O(p^2)$ and $O(p^4)$ low-energy parameters in terms of observables. On the other hand, for $r = m_s/m > 15$, the two-flavour condensate is not suppressed, due to the contribution induced by massive vacuum $\bar{s}s$ pairs. Thanks to the latter, the standard two-flavour χ PT is protected from multi-flavour instabilities and could provide a well-defined expansion scheme in powers of non-strange quark masses.

1 Introduction

Understanding chiral symmetry breaking (χ SB) in low-energy QCD still deserves both phenomenological and theoretical efforts. First, there is a growing need to identify and to separate non-perturbative QCD effects from possible manifestations of “new physics” in experimental tests of the standard model (e.g. weak matrix elements, e'/ϵ , $(g-2)_\mu, \dots$). Furthermore, the subject has its own theoretical interest. Vector-like gauge theories such as QCD formulated in a large Euclidean box allow for a particularly attractive interpretation of spontaneous χ SB in terms of the lowest modes of the Dirac operator averaged over all gluon configurations [1]. In QCD-like theories, some characteristic properties of the Dirac spectrum have been proven [2, 3] and possible consequences for chiral order parameters have been conjectured [4]. More generally, this approach to χ SB suggests an analogy with disordered systems of higher dimensionality ($d = 4$) emphasising notions such as the average and the fluctuation of the density of small Dirac eigenvalues, as well as the transport properties (e.g. conductivity) [5]. Finally, the cornerstone of the whole of this investigation is chiral perturbation theory (χ PT) [6, 7], which provides a systematic link between

theoretical characteristics of χ SB (described by order and fluctuation parameters) on one hand and observable properties of Goldstone bosons (masses, decay constants, scattering amplitudes, decay form-factors, ...) on the other hand.

During the last few years both theoretical and experimental progress was achieved along these lines. It has been suggested that order parameters of χ SB, in particular the quark–antiquark condensate $\langle \bar{q}q \rangle$, could strongly depend on the number N_f of light flavours [8, 9]. As N_f increases, $\langle \bar{q}q \rangle$ as well as the Goldstone boson coupling F_π are gradually suppressed, due to the paramagnetic behaviour of Dirac eigenvalues and to increasing fluctuations of the density of states [3, 4]. This effect is induced by light-quark loops and it cannot be detected in quenched lattice simulations. Actually, there are two kinds of paramagnetic effects generated by loops of sea quarks which are both of the same origin¹: the massless loops suppress chiral order parameters whereas the massive sea-quark pairs enhance them, as long as their mass is of order Λ_{QCD} or smaller. In nature, this last remark merely concerns the strange quark, whose mass is slightly below Λ_{QCD} . The abundance of strange quark–antiquark pairs in the vacuum can thus lead to a different behaviour of two-flavour ($m_s \sim \Lambda_{\text{QCD}}$) and three-flavour ($m_s = 0$) chiral dynamics. Such a difference would be characterised by a non-negligible vacuum correlation between strange and non-strange quark

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¹ A sea-quark loop is one with no external source attached to it

pairs, implying in turn large $1/N_c$ corrections and violation of the OZI-rule in the scalar channel. The latter is actually observed [10] and a strong variation of χ SB between $N_f = 2$ and $N_f = 3$ has been indeed reported on the basis of sum-rule studies [11, 12, 14] using as input the available information about the scalar sector 0^{++} .

Such a possibility should now be considered in the light of the new experimental information on low-energy $\pi\pi$ scattering which has been recently published [15] and analysed [16–20]. The outcome of these analyses shows that, in the presence of massive $\bar{s}s$ pairs in the vacuum, the two-flavour condensate $\langle\bar{u}u\rangle$ is large and dominates the $SU(2) \times SU(2)$ symmetry-breaking effects [19, 20]. Accordingly, the standard two-flavour χ PT expansion in powers of m_u and m_d [6] should be expected to converge rather well. On the other hand, large vacuum fluctuations of $\bar{q}q$ pairs would result into a large difference between $N_f = 2$ and $N_f = 3$ condensates, destabilising the three-flavour expansion. Indeed, a detailed $SU(3) \times SU(3)$ analysis of Goldstone boson masses and decay constants within the standard two-loop χ PT [21] has revealed an anomalously large $O(p^6)$ contribution to M_π^2 , depending on a fine tuning of the LECs $L_4(\mu)$ and $L_6(\mu)$ – which precisely reflect vacuum fluctuations. The purpose of this paper is to show that the difference in the chiral behaviour of two-flavour and multi-flavour QCD described above could be naturally explained in terms of the interplay between vacuum fluctuations (of small Dirac eigenvalues) and chiral order (described by order parameters such as $\langle\bar{q}q\rangle$).

We start by considering Ward identities and low-energy theorems for the two-point functions $\langle D^a D^b \rangle$ and $\langle V^a V^b - A^a A^b \rangle$, where V^a , A^a and D^a are (charged) vector currents, axial currents and the divergences of the latter, respectively. We write these identities in a form reminiscent of the χ PT expansion of $M_P^2 F_P^2$ and F_P^2 ($P = \pi, K, \eta$), including explicitly the leading and next-to-leading orders in powers of quark masses [7] and collecting (*not* neglecting) all remaining orders into well-defined “remainders”. We refer to Ward identities written in this way as “mass and decay constant identities”.

We then show that there exist two exact non-linear relations between the order parameters $\langle\bar{q}q\rangle$ and F_P^2 in the chiral limit $SU(N_f) \times SU(N_f)$ with $N_f \geq 3$ and two “fluctuation parameters” which are defined in terms of the standard LECs $L_6(\mu)$ and $L_4(\mu)$: in these relations, all the effects of higher χ PT orders are absorbed into a finite multiplicative renormalisation of order and fluctuation parameters. The deviation from 1 of the corresponding renormalisation constants (rescaling factors) remains under control to the extent that the NNLO remainders in mass and decay constant identities are small. The χ PT series is reproduced in the limit of small fluctuation parameters, implying a very precise fine tuning of $L_6(\mu)$ and $L_4(\mu)$. Otherwise (and in particular for large fluctuation parameters), the multi-flavour chiral condensate is suppressed, and the standard χ PT interpretation of mass and decay constant identities breaks down. This need not affect the overall convergence of the χ PT series. We actually expect that for the physical value of m_s the multi-flavour

χ PT still makes sense globally. In this case the instability caused by large fluctuations and by the suppression of the quark condensate can be cured by a simple “resummation” which amounts to replacing the perturbative solution of the non-linear relations between order and fluctuation parameters by their exact algebraic solution. This merely modifies the standard way of expressing the parameters of the effective Lagrangian in terms of observable quantities [7].

The multi-flavour mass and decay constant identities can be further used to define the two-flavour order parameters, by taking the limit $m_u, m_d \rightarrow 0$ but keeping m_s fixed at its physical value. In this way we show that the two-flavour condensate and decay constant are not affected by the large fluctuations which suppress the multi-flavour condensate. In addition, this allows one to discuss the connection with two-flavour χ PT and the new $\pi\pi$ scattering data.

The plan of this article is the following. In Sect. 2, we discuss the impact of loops of massive sea quarks (typically, $\bar{s}s$ pairs) on the pattern of χ SB. We introduce in Sect. 3 the mass and decay constant identities for a generic $N_f \geq 3$, and show that they lead to an exact system of relations between order and fluctuation parameters, presented in Sect. 4. The general properties of this system are then discussed in the light of the positivity and (conjectured) paramagnetic inequalities that the order and fluctuation parameters have to obey (Sect. 5). Section 6 is devoted to the study (in the plane of fluctuation parameters) of the critical line where the symmetry is restored, i.e. both the condensate and the decay constant vanish.

Section 7 briefly summarises properties of the large- N_c limit in which fluctuations are suppressed. Then in Sect. 8 we discuss the opposite limit of large fluctuations. A possible realisation of the latter can be interpreted as a limit of large N_f . It is shown that the limit of large fluctuations is in principle different from the symmetry restoration limit: despite the continuous vanishing of the multi-flavour quark condensate, the decay constant stays non-zero. Section 9 deals with the $SU(2) \times SU(2)$ chiral limit: we check once more that in the large-fluctuation limit the two-flavour condensate remains non-zero and the two-flavour Gell-Mann–Oakes–Renner relation is approximately obeyed, and we discuss briefly the recent results on $\pi\pi$ scattering. The conclusion and a few appendices close the paper.

2 Role of the mass of sea quarks

When defining a chiral limit or a chiral order parameter, it should be stated which fermions are taken massless and which quarks in the sea are left massive. The simplest situation is the one with just N massless fermions ψ_1, \dots, ψ_N , (i.e. $m_1 = m_2 = \dots = m_N = m \rightarrow 0$) and no other massive fermions left. The condensate of this purely massless theory is defined as

$$\delta_{ij}\sigma(N) = -\lim_{m \rightarrow 0} \langle \bar{\psi}_i \psi_j \rangle = -\langle \bar{\psi}_i \psi_j \rangle_N. \quad (1)$$

The expression of the condensate in terms of eigenvalues λ_n of the Euclidean Dirac operator $\gamma_\mu D_\mu[G]$ defined in a box $L \times L \times L \times L$ with periodic boundary conditions (up to a gauge transformation) is well known as the Banks–Casher formula [1]. Formally, it can be written as

$$\sigma(N) = \lim \frac{1}{V} \left\langle \left\langle \sum_n \frac{m}{m^2 + \lambda_n^2[G]} \right\rangle \right\rangle. \quad (2)$$

where \lim means taking $V \rightarrow \infty$ first and $m \rightarrow 0$ afterwards. The average $\langle \langle \rangle \rangle$ over Euclidean gluon configurations involves the N th power of the fermion determinant $\Delta^N(m, G)$. Since for $m \rightarrow 0$, $V \rightarrow \infty$ only the smallest eigenvalues contribute in (2), it is conceivable that the main N dependence merely arises from the infrared part of the determinant $\Delta(m, G) = \Delta_{\text{IR}} \Delta_{\text{UV}}$, defined as

$$\Delta_{\text{IR}}(m, G) = m^{|\nu|} \prod_{n < K} \frac{m^2 + \lambda_n^2}{\mu^2 + \omega_n^2}, \quad (3)$$

where ν is the winding number of the gluon configuration G and $K(\Lambda, G)$ is an integer corresponding to a cutoff Λ such that $\lambda_K = \Lambda$. Positive eigenvalues are ranked in ascending order $\lambda_1 < \lambda_2 < \dots < \lambda_K = \Lambda$. The numbers ω_n are defined by the Vafa–Witten bound [3] for the Dirac eigenvalues:

$$\lambda_n < C \left(\frac{n}{V} \right)^{1/d} \equiv \omega_n. \quad (4)$$

The existence of such a uniform bound independent of gauge field configurations is a specific property of QCD-like gauge theories. It implies $(\Delta_{\text{IR}})^{N+1} < (\Delta_{\text{IR}})^N$ for any finite cutoff K and for $m < \mu$. In the chiral limit $V \rightarrow \infty, m \rightarrow 0$ one can then expect the paramagnetic inequality

$$\sigma(N+1) \leq \sigma(N). \quad (5)$$

In the real world, the situation is slightly more complicated due to the role of massive virtual quark pairs which may be present in the vacuum. Notice that massive and massless pairs have different chiral transformation properties and do not affect the chiral structure of the vacuum in the same way. In QCD one deals with a hierarchy of quark masses:

$$m_u < m_d \ll m_s \ll m_c < m_b \ll m_t. \quad (6)$$

Some of them (u, d, s) can be considered as light compared to the scale $\Lambda_H \sim 1 \text{ GeV}$ at which the masses of the first bound states non-protected by chiral symmetry occur. $\Lambda_H \sim 4\pi F_\pi$ is the reference scale in χ PT expansions in powers of p/Λ_H . A different question can be asked in connection with the structure of the vacuum. Some quarks (c, b, t) can be considered as too heavy to form abundant vacuum pairs. In this case, the characteristic scale is not Λ_H but the lower scale Λ_{QCD} . The reason is that we are not interested in the production of massive hadrons but in the creation of massive virtual $\bar{q}q$ pairs. The latter will be most probable if the quark mass is of order Λ_{QCD} or slightly lower (if $m \rightarrow 0$, the chiral properties of the corresponding pairs tend smoothly to the massless case).

This reasoning already singles out the strange quark among all six quarks we know. Its mass $m_s \sim 160 \text{ MeV}$ (at $\mu = 1 \text{ GeV}$) [22] is sufficiently low compared to Λ_H to legitimate a $SU(3) \times SU(3)$ chiral expansion. On the other hand, m_s is sufficiently close to Λ_{QCD} to expect a significant presence of massive $\bar{s}s$ pairs in the vacuum.

We define the $SU(N_f) \times SU(N_f)$ chiral limit in QCD by taking the first N_f quarks as massless and keeping the remaining masses at their physical value. In practice, one can consider such a limit for $N_f = 2$ or for $N_f = 3$. The corresponding order parameters will be functions of the remaining non-zero masses. For instance, the two-flavour condensate is defined as

$$\Sigma(2, m_s, \dots) = - \lim_{m_u, m_d \rightarrow 0} \langle \bar{u}u \rangle, \quad (7)$$

and it is a function of m_s as well as of the heavy-quark masses denoted in (7) by the ellipsis. $u(x)$ stands for the lightest (u) quark field, and it can equivalently be replaced in (7) by the d -quark, but not by the s -quark field. The three-flavour condensate is then defined as

$$\Sigma(3, \dots) = \lim_{m_s \rightarrow 0} \Sigma(2, m_s, \dots). \quad (8)$$

One expects that the effect of heavy-quark masses m_c, m_b, m_t on $\Sigma(2)$ and $\Sigma(3)$ remains small and could be eventually estimated. For simplicity, we shall neglect all effects of heavy-quark masses in the sequel. In this approximation, $\Sigma(3)$ coincides with $\sigma(3)$ defined in (1). It is a clean probe of the chiral structure of the vacuum of QCD with nothing but three massless quarks: once one sets $m_s = 0$, there is no more massive quark left which would be sufficiently light to pollute the vacuum $|\Omega\rangle_3$.

The situation is rather different in the two-flavour chiral limit $m_u = m_d = m \rightarrow 0$, keeping the strange quark mass m_s at its physical value. Since m_s is not very large compared to Λ_{QCD} and the vacuum is polluted by massive $\bar{s}s$ pairs, it is difficult to relate $\Sigma(2)$ to the genuine condensate $\sigma(2)$ characteristic of the theory with nothing but two massless quarks. This situation occurs as long as m_s remains of order Λ_{QCD} . One can gain more insight into the m_s dependence of $\Sigma(2)$ from the formula in Euclidean space:

$$\Sigma(2) = - \frac{\langle \bar{u}u \exp[-m_s \int \bar{s}s] \rangle_3}{\langle \exp[-m_s \int \bar{s}s] \rangle_3}, \quad (9)$$

where on the right hand side the expectation value is taken with respect to the vacuum of the $SU(3) \times SU(3)$ invariant theory. It is seen that in the absence of correlations between the strange and non-strange $\bar{q}q$ pairs, (9) implies $\Sigma(2) = \Sigma(3)$, in agreement with large- N_c expectations. For $m_s \ll \Lambda_H$, (9) can be rewritten as

$$\Sigma(2) = \Sigma(3) + m_s \bar{Z}_1^s + \dots, \quad (10)$$

where \bar{Z}_1^s denotes the connected correlator of $\bar{s}s$ and $\bar{u}u$ pairs:

$$\bar{Z}_1^s = \lim_{m \rightarrow 0} \int dx \langle \bar{s}s(x) \bar{u}u(0) \rangle_{\text{con}}, \quad (11)$$

Notice that in the limit $m_{u,d} \rightarrow 0$, $\bar{u}u$ can be equally replaced by $(\bar{u}u + \bar{d}d)/2$. The correlator (11) measures the violation of the OZI rule in the isoscalar scalar (i.e. vacuum) channel and it can be estimated using the experimental information now available in this channel [11,12,14]. It is related to the standard $O(p^4)$ LEC $L_6(\mu)$ and it turns out to be larger than what is expected on the basis of large- N_c considerations [7].

It is useful to express $\Sigma(2)$, $\Sigma(3)$ and the correlator $\langle\langle \bar{s}s(\bar{u}u) \rangle\rangle$ in terms of the eigenvalues λ_n of the Euclidean Dirac operator. Neglecting heavy quarks, both $\Sigma(3)$ and $\Sigma(2)$ concern the theory with the same total number of fermions: in $\Sigma(3)$, one sets $m_u = m_d = m_s = m \rightarrow 0$, whereas in $\Sigma(2)$ $m_u = m_d = m \rightarrow 0$ but $m_s \sim \Lambda_{\text{QCD}}$ is held fixed. Hence, the corresponding Banks–Casher formula becomes [1]

$$\Sigma(N_f) = \lim \frac{1}{V} \left\langle\left\langle \sum_n \frac{m}{m^2 + \lambda_n^2} \right\rangle\right\rangle_{N_f}, \quad (12)$$

where the only difference lies in the determinant inserted in the average over gluon configurations: $\Delta^3(m, G)$ for $\Sigma(3)$ and $\Delta^2(m, G)\Delta(m_s, G)$ in the case of $\Sigma(2)$. Comparing the corresponding infrared parts (3) which are expected to dominate in (12), one has

$$\begin{aligned} & m^{3|\nu|} \prod_{n < K} \left(\frac{m^2 + \lambda_n^2}{\mu^2 + \omega_n^2} \right)^3 \\ & \leq m^{2|\nu|} m_s^{|\nu|} \prod_{n < K} \frac{(m^2 + \lambda_n^2)^2 (m_s^2 + \lambda_n^2)}{(\mu^2 + \omega_n^2)^3}, \end{aligned} \quad (13)$$

as long as $m \rightarrow 0$ and $m_s \sim \Lambda_{\text{QCD}}$. This suggests that the paramagnetic inequality (5) holds even in the presence of massive strange quarks in the sea:

$$\Sigma(2) \geq \Sigma(3). \quad (14)$$

The conjectured inequality (14) will play an important role in the sequel.

It remains to express the correlator (11) in terms of Dirac eigenvalues; see e.g. [4]. One has

$$\bar{Z}_1^s = \lim \frac{1}{V} \left\langle\left\langle \sum_{nk} \frac{m}{m^2 + \lambda_n^2} \frac{m_s}{m_s^2 + \lambda_k^2} \right\rangle\right\rangle_{\text{con}}. \quad (15)$$

For small m and m_s , only small Dirac eigenvalues contribute and the expression (15) measures their correlations. For $m_s \sim m$, \bar{Z}_1^s describes the fluctuations of the density of states $\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n)$ near $\lambda \sim 0$. The positivity of (15) is in agreement with the paramagnetic inequality (14). In this paper, we investigate the possibility that multi-flavour QCD vacuum behaves as a strongly correlated fermion–anti-fermion system characterised by large fluctuations \bar{Z}_1^s of the density of small Dirac eigenvalues.

Let us mention without proof that the present discussion can be easily extended to the order parameter:

$$F^2(N_f) = \lim_{m_1 \dots m_{N_f} \rightarrow 0} F_\pi^2, \quad (16)$$

defined as before by taking the first N_f quarks massless and leaving the remaining quarks at their physical masses. The dependence of these order parameters on N_f and on the sea-quark masses is qualitatively similar to that of the quark condensates.

3 $SU(N_f) \times SU(N_f)$ chiral symmetry for $N_f \geq 3$

We consider QCD with $N_f = n + 2$ light flavours which is very like our actual three-flavour QCD equipped with n copies of the strange quark degenerate in mass: u and d quarks with ultralight degenerate masses $m_u = m_d = m$ and n “strange” copies $s_1 \dots s_n$ with a common mass $m_s \gg m$ but still light compared to the QCD scales.

In the chiral limit $m = m_s = 0$, the $SU(2+n) \times SU(2+n)$ chiral symmetry is assumed to be spontaneously broken down to the diagonal subgroup $SU_V(n+2)$. This symmetry is explicitly broken by the mass difference $m_s - m$ to $SU(2) \times SU(n)$, which is the exact symmetry of our problem. We have $(n+2)^2 - 1 = n^2 + 4n + 3$ pseudo-Goldstone bosons: 3 pions of mass M_π , $4n$ kaons ($\bar{u}s_i, \bar{d}s_i$ for $i = 1 \dots n$, plus conjugates) of common mass M_K , the η -meson with structure $\lambda_\eta = (1+2/n)^{-1/2} \text{diag}[1, 1; -2/n \dots -2/n]$ and mass M_η , and finally the extra $n^2 - 1$ $\bar{s}_i s_j$ states whose mass will be denoted as M_X .

This framework will be used to discuss the response of order parameters of the chiral symmetry $SU_L(N_f) \times SU_R(N_f)$ to large vacuum fluctuations both for $N_f \geq 3$ and for $N_f = 2$. We will see that these two cases behave rather differently. It will prove useful to keep a generic value for N_f , although our main interest will be on the case $N_f = 3$ and all our conclusions apply to this case.

3.1 Multi-flavour mass and decay constant identities

For all n we are using the same $O(p^4)$ effective Lagrangian as for the $n = 1$ case of [7] (except for one additional invariant $\langle \nabla^\mu U^\dagger \nabla^\nu U \nabla_\mu U^\dagger \nabla_\nu U \rangle$, irrelevant for our purpose). The LECs have an a priori unknown dependence on n ; therefore we keep the same notation for all LECs with an extra index n . This is justified to the extent that the symmetry properties are as described above.

The chiral expansion of Goldstone boson masses is very close to the case $n = 1$ discussed in [14]:

$$\begin{aligned} F_\pi^2 M_\pi^2 &= 2m\Sigma(n+2) + 2m(nm_s + 2m)Z_n^s + 4m^2 A_n \\ &+ 4m^2 B_{0n}^2 L + F_\pi^2 M_\pi^2 d_{\pi n}, \end{aligned} \quad (17)$$

$$\begin{aligned} F_K^2 M_K^2 &= (m_s + m)\Sigma(n+2) + (m_s + m)(nm_s + 2m)Z_n^s \\ &+ (m_s + m)^2 A_n + m(m_s + m)B_{0n}^2 L \\ &+ m_s(m_s + m)B_{0n}^2 L'_n + F_K^2 M_K^2 d_{Kn}, \end{aligned} \quad (18)$$

$$\begin{aligned} F_X^2 M_X^2 &= 2m_s\Sigma(n+2) + 2m_s(nm_s + 2m)Z_n^s + 4m_s^2 A_n \\ &+ 4m_s^2 B_{0n}^2 L'_n + F_X^2 M_X^2 d_{Xn}, \end{aligned} \quad (19)$$

and likewise for $F_\eta^2 M_\eta^2$ (see Appendix A). The connection with the standard LECs of the $N_f \geq 3$ effective Lagrangian is

$$Z_n^s = 32B_{0n}^2 \left\{ L_6^n(\mu) - \frac{1}{512\pi^2} \left[\log \frac{M_K^2}{\mu^2} + \frac{2}{(n+2)^2} \log \frac{M_\eta^2}{\mu^2} \right] \right\}, \quad (20)$$

$$A_n = 16B_{0n}^2 \left\{ L_8^n(\mu) - \frac{n}{512\pi^2} \left[\log \frac{M_K^2}{\mu^2} + \frac{2}{n+2} \log \frac{M_\eta^2}{\mu^2} \right] \right\}. \quad (21)$$

We have the ratio

$$B_{0n} = \frac{\Sigma(n+2)}{F^2(n+2)}, \quad (22)$$

and $\Sigma(n+2)$ and $F(n+2)$ denote the condensate and the pseudoscalar decay constant for all $n+2$ massless quarks. The remainders $F_P^2 M_P^2 d_P^2$ collect all higher-order terms, starting at the next-to-next-to-leading order $O(m_q^3)$ (NNLO) [hence $d_n = O(m_q^2)$]. The (n -independent) combination of chiral logarithms L has the same meaning as in [14]:

$$L = \frac{1}{32\pi^2} \left[3 \log \frac{M_K^2}{M_\pi^2} + \log \frac{M_\eta^2}{M_K^2} \right]. \quad (23)$$

whereas L' contains all chiral logarithms involving the unphysical mass M_X :

$$L'_n = \frac{n-1}{16\pi^2 n} \left\{ \log \frac{M_\eta^2}{M_K^2} - (n+1) \log \frac{M_X^2}{M_K^2} \right\}, \quad (24)$$

which vanishes for $n=1$ as it should. Similar expressions are derived for the decay constants:

$$F_\pi^2 = F^2(n+2) + 2(nm_s + 2m)\tilde{\xi}_n + 2m\xi_n + \frac{2mB_{0n}}{16\pi^2} \left\{ \log \frac{M_\eta^2}{M_K^2} + 2 \log \frac{M_K^2}{M_\pi^2} \right\} + F_\pi^2 e_{\pi n}, \quad (25)$$

$$F_K^2 = F^2(n+2) + 2(nm_s + 2m)\tilde{\xi}_n + (m_s + m)\xi_n + mB_{0n}L + m_s B_{0n}L'_n + F_K^2 e_{Kn}, \quad (26)$$

$$F_X^2 = F^2(n+2) + 2(nm_s + 2m)\tilde{\xi}_n + 2m_s\xi_n + \frac{2m_s B_{0n}}{16\pi^2} \left\{ \log \frac{M_\eta^2}{M_K^2} - n \log \frac{M_X^2}{M_K^2} \right\} + F_X^2 e_{Xn}. \quad (27)$$

while F_η is given in Appendix A. The scale-invariant constants ξ_n and $\tilde{\xi}_n$ are related to the LECs L_4 and L_5 as follows:

$$\tilde{\xi}_n = 8B_{0n} \left\{ L_4^n(\mu) - \frac{1}{256\pi^2} \log \frac{M_K^2}{\mu^2} \right\}, \quad (28)$$

$$\xi_n = 8B_{0n} \left\{ L_5^n(\mu) - \frac{1}{256\pi^2} \left[n \log \frac{M_K^2}{\mu^2} + 2 \log \frac{M_\eta^2}{\mu^2} \right] \right\}. \quad (29)$$

The remainders $F_P^2 e_{Pn}$ collect the higher-order terms, starting at NNLO [$O(m_q^2)$].

3.2 Low-energy constants

It is a simple exercise of algebra to combine (17) and (18), and (25) and (26), to get the expression of the LECs:

$$\frac{2m}{F_\pi^2 M_\pi^2} [\Sigma(n+2) + (nr+2)mZ_n^S] = 1 - \epsilon(r) - d_n - \frac{4m^2 B_{0n}^2}{F_\pi^2 M_\pi^2} \frac{r}{r-1} (L - L'_n), \quad (30)$$

$$\frac{4m^2 A_n}{F_\pi^2 M_\pi^2} = \epsilon(r) + d'_n + \frac{4m^2 B_{0n}^2}{F_\pi^2 M_\pi^2} \frac{1}{r-1} (L - rL'_n), \quad (31)$$

with

$$\epsilon(r) = 2 \frac{r_2 - r}{r^2 - 1}, \quad r_2 = 2 \left(\frac{F_K M_K}{F_\pi M_\pi} \right)^2 - 1. \quad (32)$$

d_n and d'_n are linear combinations of the remainders $d_{\pi n}$ and d_{Kn} :

$$d_n = \frac{r+1}{r-1} d_{\pi n} - \left(\epsilon(r) + \frac{2}{r-1} \right) d_{Kn}, \quad d'_n = d_n - d_{\pi n}. \quad (33)$$

A similar work can be performed for the decay constants:

$$\begin{aligned} \frac{2m\xi}{F_\pi^2} &= \eta(r) + e'_n - \frac{2mB_{0n}}{(r-1)F_\pi^2} \left[L + rL'_n - \frac{1}{8\pi^2} \left(\log \frac{M_\eta^2}{M_K^2} + 2 \log \frac{M_K^2}{M_\pi^2} \right) \right], \\ (nr+2) \frac{2m\tilde{\xi}}{F_\pi^2} &= 1 - \eta(r) - e_n - \frac{F^2(n+2)}{F_\pi^2} + \frac{2mB_{0n}}{(r-1)F_\pi^2} \left[L + rL'_n - \frac{r+1}{16\pi^2} \left(\log \frac{M_\eta^2}{M_K^2} + 2 \log \frac{M_K^2}{M_\pi^2} \right) \right], \end{aligned} \quad (34)$$

with

$$\eta(r) = \frac{2}{r-1} \left(\frac{F_K^2}{F_\pi^2} - 1 \right), \quad (36)$$

$$e_n = \frac{r+1}{r-1} e_{\pi n} - \left(\eta(r) + \frac{2}{r-1} \right) e_{Kn} \equiv e'_n + e_{\pi n}. \quad (37)$$

We recall that $\epsilon(r)$ and $\eta(r)$ are suppressed at large values of r (≥ 15). We expect then $d'_n \ll d_n \sim d_{\pi n}$ and $e'_n \ll e_n \sim e_{\pi n}$.

One easily checks that all the formulae displayed in this section reduce to the mass and decay constant identities obtained in [14] in the case $n=1$.

3.3 $SU(2) \times SU(2)$ order parameters

In the $SU(n+2) \times SU(n+2)$ chiral limit ($m = m_s = 0$), we define the dimensionless order parameters

$$X(n+2) = \frac{2m\Sigma(n+2)}{F_\pi^2 M_\pi^2},$$

$$\begin{aligned} Y(n+2) &= \frac{2mB_{0n}}{M_\pi^2}, \\ Z(n+2) &= \frac{F^2(n+2)}{F_\pi^2}. \end{aligned} \quad (38)$$

The Gell-Mann–Oakes–Renner ratio $X(n+2)$ measures the quark condensate in physical units, while $Z(n+2)$ does the same for the decay constant. We have $X(n+2) = Y(n+2)Z(n+2)$.

We consider now the $SU(2) \times SU(2)$ chiral limit where only m vanishes (and the n copies of the strange quark remain massive), in order to investigate the effect of massive sea quarks on two-flavour chiral dynamics. We define the quark condensate and the decay constant in this limit as

$$\Sigma(2) = \lim_{m \rightarrow 0} \frac{F_\pi^2 M_\pi^2}{2m}, \quad F^2(2) = \lim_{m \rightarrow 0} F_\pi^2, \quad (39)$$

and the dimensionless order parameters

$$\begin{aligned} X(2) &= \frac{2m\Sigma(2)}{F_\pi^2 M_\pi^2}, \\ Z(2) &= \frac{F^2(2)}{F_\pi^2}, \\ Y(2) &= \frac{X(2)}{Z(2)} = \frac{2mB(2)}{M_\pi^2}. \end{aligned} \quad (40)$$

From (17) and (30), we derive the expression for the two-flavour condensate:

$$\begin{aligned} X(2)[1 - \bar{d}_{\pi n}] &= \frac{nr}{nr+2} \\ &\times [1 - \epsilon(r) - d_n - [Y(n+2)]^2 f_n] \\ &+ \frac{2}{nr+2} X(n+2), \end{aligned} \quad (41)$$

where a quantity topped with a bar is considered in the limit $m \rightarrow 0$ and

$$\begin{aligned} f_n &= \frac{M_\pi^2}{F_\pi^2} \left(\frac{r}{r-1} [L - L'_n] - \frac{nr+2}{2} \Delta Z_n^s \right), \\ \bar{Z}_n^s &= Z_n^s + B_{0n}^2 \Delta Z_n^s, \\ \Delta Z_n^s &= \frac{1}{16\pi^2} \left[\log \frac{M_K^2}{M_K^2} + \frac{2}{(n+2)^2} \log \frac{M_\eta^2}{M_\eta^2} \right]. \end{aligned} \quad (42)$$

In a similar fashion, the two-flavour decay constant can be read from (25) and (35):

$$\begin{aligned} Z(2)[1 - \bar{e}_{\pi n}] &= \frac{nr}{nr+2} [1 - \eta(r) - e_n - Y(n+2)g_n] \\ &+ \frac{2}{nr+2} Z(n+2), \end{aligned} \quad (44)$$

with the quantities

$$\begin{aligned} g_n &= \frac{M_\pi^2}{F_\pi^2} \left(\frac{r}{r-1} [L - L'_n] + \frac{r+1}{r-1} \frac{1}{32\pi^2} \log \frac{M_\eta^2}{M_\pi^2} \right. \\ &\left. - (nr+2) \Delta \tilde{\xi}_n^s \right). \end{aligned} \quad (45)$$

$$\tilde{\xi}_n = \tilde{\xi}_n + B_{0n} \Delta \tilde{\xi}_n, \quad \Delta \tilde{\xi}_n = \frac{1}{32\pi^2} \log \frac{M_K^2}{M_K^2}. \quad (46)$$

f_n and g_n contain chiral logarithms of two different types and signs: the first involve the masses of the Goldstone bosons $M_{\pi,K,\eta,X}$ and are positive, while the latter are negative and combine ratios of masses of the same meson, considered in the massive theory and in the $SU(2) \times SU(2)$ chiral limit ($m \rightarrow 0$). We can estimate f_n and g_n by iterating the previous mass and decay constant identities as explained in Appendix A and B.

4 Connection between fluctuation and order parameters

We want now to investigate the connection between chiral order parameters and vacuum fluctuations. For a generic $n > 0$, the mass and decay constant identities yield a very simple expression of the two order parameters of fundamental interest, namely the condensate $\Sigma(n+2)$ and the decay constant $F(n+2)$, in terms of the two OZI-rule violating LECs L_4 and L_6 . These two large- N_c suppressed constants directly reflect vacuum fluctuations of $\bar{q}q$ pairs and they provide a convenient framework to discuss completely and transparently how these fluctuations affect the order parameters.

The first step consists in eliminating the LECs A_n and ξ_n from the mass and decay constant identities for the pion and the kaon (17) and (18) and (25) and (26), leading to (30) and (35). Then we can reexpress the (OZI-rule violating) constants $\tilde{\xi}_n$ and Z_n^s in terms of L_4 and L_6 , and obtain the two desired relations between the order parameters X, Y, Z and the fluctuation parameters L_4 and L_6 :

$$X(n+2) = 1 - \epsilon(r) - d_n - [Y(n+2)]^2 \rho_n/4, \quad (47)$$

$$Z(n+2) = 1 - \eta(r) - e_n - Y(n+2)\lambda_n/4. \quad (48)$$

The constants L_6 and L_4 enter the discussion through the combinations

$$\rho_n = 64 \frac{M_\pi^2}{F_\pi^2} (nr+2) \Delta L_6^n, \quad \lambda_n = 32 \frac{M_\pi^2}{F_\pi^2} (nr+2) \Delta L_4^n, \quad (49)$$

where the scale-independent differences $\Delta L_i^n = L_i^n(\mu) - L_i^{n,\text{crit}}(\mu)$ involve the critical values of the LECs defined as

$$\begin{aligned} L_4^{n,\text{crit}}(\mu) &= \frac{1}{256\pi^2} \log \frac{M_K^2}{\mu^2} \\ &- \frac{r}{8(r-1)(nr+2)} [L - L'_n] \\ &- \frac{1}{256\pi^2} \frac{r+1}{(r-1)(nr+2)} \log \frac{M_\eta^2}{M_\pi^2}, \end{aligned} \quad (50)$$

$$\begin{aligned} L_6^{n,\text{crit}}(\mu) &= \frac{1}{512\pi^2} \left(\log \frac{M_K^2}{\mu^2} + \frac{2}{(n+2)^2} \log \frac{M_\eta^2}{\mu^2} \right) \\ &- \frac{r}{16(nr+2)(r-1)} [L - L'_n]. \end{aligned} \quad (51)$$

Comparing the fluctuation parameters λ_n and ρ_n (49) to 1 provides a quantitative measure of the effect of the

LECs L_4 and L_6 on observable quantities. The effect disappears if L_4 and L_6 are fine tuned to their critical values (50) and (51), which for $n = 1$, $r = 25$, $\mu = M_\rho$ become $L_4^{\text{crit}}(M_\rho) = -0.51 \cdot 10^{-3}$ and $L_6^{\text{crit}}(M_\rho) = -0.26 \cdot 10^{-3}$.

Notice that even a small deviation from the critical values is amplified by the large numerical coefficients in (49). It is possible and convenient to absorb the NNLO and higher-order contributions – represented in (47) and (48) by the remainders d_n and e_n – into a multiplicative renormalisation of order and fluctuation parameters. Defining the rescaled parameters as

$$z_n = \frac{F^2(n+2)}{F_\pi^2[1 - \eta(r) - e_n]}, \quad y_n = \frac{1 - \eta(r) - e_n}{1 - \epsilon(r) - d_n} \frac{2mB_{0n}}{M_\pi^2}, \quad (52)$$

$$u_n = \lambda_n k_n, \quad v_n = \rho_n k_n, \quad k_n(r) = \frac{1 - \epsilon(r) - d_n}{(1 - \eta(r) - e_n)^2}, \quad (53)$$

the two basic equations inferred from the mass and decay constant identities take now the concise form

$$z_n + \frac{1}{4}u_n y_n = 1, \quad (54)$$

$$z_n + \frac{1}{4}v_n y_n = \frac{1}{y_n}. \quad (55)$$

In addition, the multi-flavour quark condensate [expressed in GOR units, cf. (38)] can be rewritten in terms of y_n and z_n :

$$X(n+2) = [1 - \epsilon(r) - d_n]y_n z_n. \quad (56)$$

At this place, a few remarks are in order.

(i) The above analysis holds for a generic $n > 0$, including the physical case $n = 1$ of three flavours. On the other hand, the case of two massless flavours ($m = 0$, m_s non-zero) requires a separate discussion based on (41) and (44) above.

(ii) The relations (54) and (55) between the (rescaled) order parameters y, z on one hand and the fluctuation parameters u, v on the other hand represent exact identities which do not result from any approximation or expansion. The influence of higher chiral orders – NNLO and beyond – is entirely encoded in the rescaling factors (52) and (53) through the remainders d_n and e_n . The latter, defined in (33) and (37), stem from $O(p^6)$ (and higher) contributions to the mass and decay constant identities. They are both of order $d_n, e_n = O(m_q^2)$ and should remain small unless the whole χ PT series blows up.

(iii) We expect the rescaling factors in (52) and (53) to be close to 1. Two circumstances could spoil this expectation. First, if the quark mass ratio r were small (typically $r < 10$), the quantity $\epsilon(r)$ would get close to 1. On the other hand, for $r > 15$ one has $\epsilon(r) < 0.2$ and $\eta(r) < 0.07$. Such higher values of r are preferred by the new low-energy $\pi\pi$ scattering data [20], discussed in Sect. 9. Second, even if the remainders d_n and e_n are small in the physical case $n = 1$ (say 10% or less), their size could

grow when n increases. We shall come back to the large- n behaviour of the theory in Sect. 8. In either case, we would not be allowed to treat perturbatively the rescaling factors as close to 1.

(iv) Let us consider a solution of the generic system (54) and (55) giving the order parameters y and z in terms of the fluctuation parameters u and v . We can then read from (31) and (34) together with (21) and (29) the following parameters of \mathcal{L}_{eff} : $2mB_{0n}$, $F_0^2 = F^2(n+2)$, $L_5(\mu)$ and $L_8(\mu)^2$ as functions of $r = m_s/m$, $L_4(\mu)$, $L_6(\mu)$, the NNLO remainders $d_{\pi n}$, $d_{K n}$ and $e_{\pi n}$, $e_{K n}$ ³. These expressions are then used in the study of other observables (e.g. Goldstone boson scattering amplitudes) to eliminate $O(p^2)$ and $O(p^4)$ LECs from the bare χ PT formulae. In particular, this procedure should be used when the NNLO remainders are (iteratively) matched with two-loop χ PT expressions. The procedure described above – eliminating constants of \mathcal{L}_{eff} in favour of observables – differs from the one used in standard χ PT, unless the fluctuation parameters u and v are small compared to 1. This might in particular affect the outcome of standard analyses beyond one loop [21].

(v) If $u_n, v_n \ll 1$, the standard multi-flavour χ PT may be recovered in two steps. First, one constructs the perturbative solution of the non-linear system (54) and (55) in powers of $u_n, v_n = O(m_q)$:

$$y_n = 1 + \frac{1}{4}(u_n - v_n) + \frac{1}{8}(u_n - v_n)^2 + \dots, \quad (57)$$

$$z_n = 1 - \frac{1}{4}u_n - \frac{1}{16}u_n(u_n - v_n) + \dots \quad (58)$$

Next, one returns to the original variables λ_n, ρ_n in (49) and to $2mB_{0n}/M_\pi^2$, $F^2(n+2)/F_\pi^2$ expanding the rescaling factors in (52) and (53) around 1 in powers of $\epsilon(r) = O(m_q)$, $\eta(r) = O(m_q)$ and of the remainders d_n , $e_n = O(m_q^2)$. Matching the latter with the explicit two-loop contributions to (17) and (18) and (25) and (26), one reproduces the standard χ PT expansion up to and including $O(p^6)$ order.

(vi) In the physical situation of three massless flavours ($n = 1$), the fluctuation parameters need not be small enough to allow the power expansion of (57) and (58): using the estimates for $L_6^1(M_\rho) = 0.6 \cdot 10^{-3}$ (central value) based on sum rules for the correlator $\langle(\bar{u}u + \bar{d}d)\bar{s}s\rangle$ [11, 12, 14] and the recent determination of $L_4^1(M_\rho) = 0.1 \cdot 10^{-3}$ (central value) from πK scattering data [23], one obtains the rough estimate $u_1 \sim 1.2$, $v_1 \sim 3.4$ (for $r = 25$). As pointed out in [4], important vacuum fluctuations of $\bar{q}q$ pairs suppress the three-flavour condensate and destabilise the χ PT expansion.

This phenomenon is a particular consequence of (54) and (55). Multiplying the latter by y and using (56), one obtains the relation

² If we include the identities for η , we can also express the constant L_7 – see Appendix A

³ Alternatively, we can express all $O(p^4)$ symmetry-breaking LECs as functions of r and the two order parameters $X(n+2)$ and $Z(n+2)$, as discussed in [14] for $n = 1$

$$X(n+2) = 2 \frac{1 - \epsilon(r) - d_n}{1 + \sqrt{1 + v_n/z_n^2}}, \quad (59)$$

which is identical (in the physical case $n = 1$) to (9) of [13].

To cope with possibly large values of the fluctuation parameters u and v , it may very well be sufficient to replace the perturbative solution (57) and (58) of the system (54) and (55) by its exact algebraic solution, and to keep the perturbative expansion of the rescaling factors in (52) and (53). Let us emphasise that the divergence of the power series (57) and (58) is a question logically disconnected from the convergence of the expansion of rescaling factors around 1: the former is related to a possible suppression of the multi-flavour condensate and of the corresponding leading order of χ PT, whereas the latter is more a question of a global convergence of the χ PT series starting at NNLO order.

In the remaining sections of this paper we concentrate on the non-perturbative analysis of the system (54) and (55) and its consequences for the breaking of both $SU(n+2) \times SU(n+2)$ and $SU(2) \times SU(2)$ chiral symmetries in various limits.

5 Positivity and paramagnetic constraints

In this section we investigate the system of equations (54) and (55) in the light of the positivity of the order parameters $X(n+2)$, $Y(n+2)$ and $Z(n+2)$ and of the conjectured paramagnetic inequalities they have to satisfy [4]. We are mainly interested in the domain allowed by these constraints in the plane of the fluctuation parameters u and v . For simplicity, we omit further reference to n and r , since the latter enter the system (54) and (55) only via rescaling factors. The system admits two solutions for the ratio of the order parameter y (related to B_{0n}):

$$y_+ = \frac{2}{1 + \sqrt{1 + v - u}}, \quad y_- = \frac{2}{1 - \sqrt{1 + v - u}}. \quad (60)$$

These two solutions depend on the difference of the two rescaled fluctuation parameters u (function of the LEC L_4) and v (function of L_6). Notice that $v - u$ is related to the particular combination $2L_6 - L_4$. The square root in (60) signals a non-perturbative resummation of vacuum fluctuations as discussed in the previous section. For $|v - u| \ll 1$, the Taylor expansion of y_+ reproduces the χ PT series, whereas y_- is of truly non-perturbative nature.

The two branches y_+ and y_- can be considered as two different sheets of the two-valued function $y(v, u)$. These sheets are tangent along the line $v - u = -1$ that coincides with the boundary of the definition domain $v > u - 1$. Along this boundary, y has the constant value 2. The two branches y_+ and y_- are drawn in Fig. 1 as functions of $v - u$. When $v - u$ increases, y_+ decreases and vanishes asymptotically at infinity, whereas y_- increases and tends to infinity for $v - u \rightarrow 0^-$.

The system (54) and (55) yields the ratio y and the (rescaled) decay constant z , both of which should be positive for an acceptable solution: the vacuum stability requires $x = yz > 0$ and z is related to $F^2(n+2)/F_\pi^2$. As far

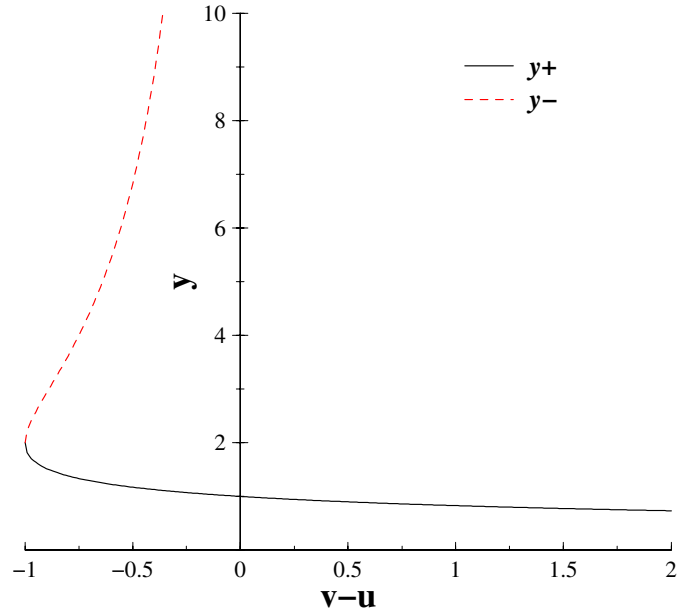


Fig. 1. The two branches for y as functions of $v - u$. The solid (black) branch is y_+ , while the dashed (red) branch is y_- .

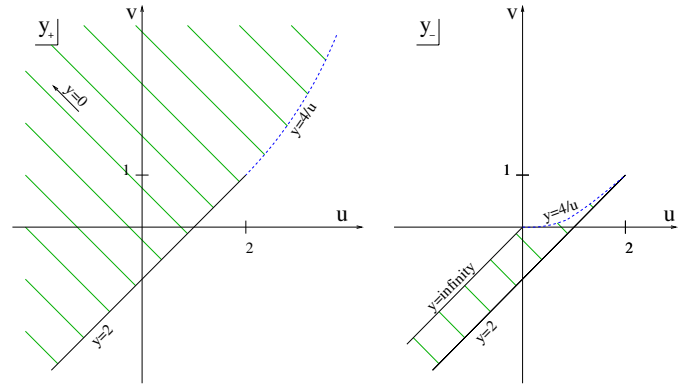


Fig. 2. The two sheets for y in the (u, v) plane: on the left, the y_+ sheet, on the right, the y_- sheet. The hatched (green) regions are the allowed domains where y and z are both positive. The value of y at each boundary is indicated. The critical line $z = 0$ of symmetry restoration is denoted with a dashed (blue) line

as y is concerned, y_+ is positive in the whole half-plane $v - u > -1$ (where $0 < y < 2$), and y_- is positive inside the strip $-1 < v - u < 0$ (where $y > 2$). The positivity of z , i.e. the condition $z = 1 - uy/4 > 0$, yields additional constraints in the (u, v) plane, especially for $u > 0$. The critical line $z = 0$ is the parabola $v = u^2/4$, along which $y = 4/u$. The condition $z > 0$ is trivially satisfied for negative u , but for $u > 0$ it leads on both sheets to (different) non-trivial bounds.

As a result the whole domain of positivity of both y and z is obtained. On the y_+ sheet, one must have $v > u - 1$ for $u < 2$, and $v > u^2/4$ for $u > 2$. On the y_- sheet, the positivity domain amounts to the part of the strip $u > v > u - 1$ situated below the parabola $v = u^2/4$. These two domains are represented in Fig. 2.

We now arrive at the constraints imposed by the paramagnetic inequalities between order parameters discussed in [4]. It was suggested that the chiral order parameters that are dominated by the lowest eigenvalues of the Euclidean Dirac operator are particularly sensitive to a paramagnetic suppression arising from the infrared part of the fermion determinant – i.e. from the light-quark loops. This applies in particular to the decay constant F^2 and to the quark condensate Σ : the more flavours from the total of $N_f = n + 2$ become massless, the more suppressed the fundamental order parameters get. As a corollary, $\Sigma(2)$ should be an increasing function of the strange quark mass m_s , as long as the latter is comparable to the QCD scale. This suggests the two paramagnetic constraints

$$X(2) \geq X(n+2), \quad Z(2) \geq Z(n+2), \quad (61)$$

which can be reexpressed using the expressions of $X(2)$ in (41) and $Z(2)$ in (44). They can be further simplified using the notation of the previous section, and yield for X and Z respectively

$$X : v \geq \frac{4}{1-D} \left[kf - \frac{D}{y^2} \right], \quad Z : u \geq \frac{4}{1-E} \left[kg - \frac{E}{y} \right], \quad (62)$$

where the NNLO remainders are rescaled: $D_n = \bar{d}_{\pi n} \cdot (nr+2)/(nr)$, $E_n = \bar{e}_{\pi n} \cdot (nr+2)/(nr)$, and f and g are positive combinations of the chiral logarithms defined in (42) and (45) and estimated in Appendix B.

The paramagnetic inequalities (61) yield therefore lower bounds for the fluctuation parameters u and v . These bounds are only useful if the ratio of order parameters y is large enough, for instance on the second sheet where y could grow. But we do not expect (62) to have much relevance, say, on the first sheet for large u and v . Using the system of (54) and (55), we may convert these lower bounds on the fluctuation parameters into upper bounds for y . The constraints on X and Z yield respectively

$$X : y \leq \frac{2}{(1-D)z + \sqrt{(1-D)^2 z^2 + 4kf}}, \quad (63)$$

$$Z : y \leq \frac{1-z(1-E)}{kg}.$$

From (63) and the discussion of the previous section, we obtain the important result that y is necessarily non-vanishing and finite on the physical domain of the two sheets:

$$0 < y \leq \min \left(\frac{1}{\sqrt{kf}}, \frac{1}{kg} \right). \quad (64)$$

Moreover, if y is large enough, (62) shows that u is positive, and thus $z = 1 - uy/4 < 1$. The paramagnetic inequalities in (61) lead therefore to bounds for the two main $SU(n+2) \times SU(n+2)$ chiral order parameters X and Z .

In the case $n = 1$ and $r = 25$, the estimation procedure detailed in Appendix B leads to Fig. 3 for u and v . We see that only the upper right part of the (u, v) plane survives. In particular, only a small fraction of the second sheet, far

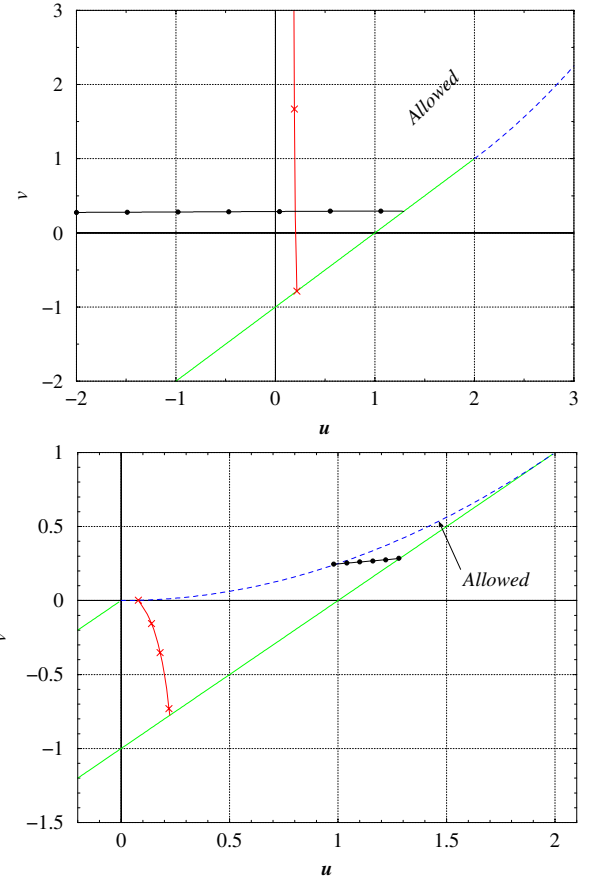


Fig. 3. Illustration of the paramagnetic inequalities on the two sheets for $n = 1$ and $r = 25$ (upper: first sheet y_+ , lower: second sheet y_-). The boundaries of the two sheets are indicated with solid (green) lines, while the (blue) dashed line is the critical parabola $z = 0$. The (black) line with filled circles indicates the lower bound for v , derived from the paramagnetic inequality for X . The (red) line with crosses indicates the lower bound for u , derived from the paramagnetic inequality for Z . The NNLO remainders D and E are set to 0

from the origin, remains available. Let us emphasise that the paramagnetic inequalities yield such constraints only if the two combinations of chiral logarithms f and g are positive.

It is worth noting that the existence of minimal values of the fluctuation parameters u and v does not contradict χ PT, which requires $u, v \rightarrow 0$ in the chiral limit. The minimal values (62) are indeed proportional to M_π^2 as the parameters u and v themselves, and therefore constrain only how quickly u and v vanish in the chiral limit.

As a conclusion, the system (54) and (55) admits two solutions for the ratio of order parameters y : they can be considered as the two sheets of a two-valued function $y(v, u)$. In the (u, v) plane of fluctuation parameters, the domain of definition of the two sheets is restricted by positivity constraints [vacuum stability, $F^2(n+2) \geq 0$] and by paramagnetic inequalities [$X(n+2) \leq X(2)$, $Z(n+2) \leq Z(2)$]. As a result, the first sheet is limited to positive (and possibly very large) values of the fluctuation parameters

u, v , whereas only a tiny region of the second sheet, far from the origin but below $u = 2, v = 1$, is allowed.

6 The case of symmetry restoration

We investigate now the vicinity of the critical line where the $SU(n+2) \times SU(n+2)$ chiral symmetry is restored. Since we exploit results from the effective Lagrangian describing the spontaneously broken phase in terms of Goldstone bosons, we can comment only on the approach to the chirally symmetric phase from the broken one (and not on specific properties of the former phase, where the $SU(n+2) \times SU(n+2)$ chiral symmetry is restored).

The symmetry restoration is equivalent to the vanishing of the decay constant F_π in the chiral limit, viz. $Z(n+2) = 0$. In the (u, v) plane, the condition $z = 0$ corresponds to the critical line $v = u^2/4$. The limit $z \rightarrow 0^+$ can be reached on both sheets: on the y_+ sheet the critical line is approached from above and $u > 2$, while on the y_- sheet, the parabola $v = u^2/4$ must be approached from below and $0 < u < 2$. On both sheets, the ratio of order parameters has the value $y = 4/u$ on the critical line.

We see that the vicinity of the origin in the (u, v) plane on the second sheet plays a special role in the discussion: y diverges there. The paramagnetic inequalities (64) however prevent us from reaching the vicinity of $u = v = 0$ on the second sheet: y is bounded and cannot become arbitrarily large. The ratio of order parameters $Y(n+2)$ remains thus finite when the critical line $Z(n+2) = 0$ is approached, so that the quark condensate $X(n+2) = Y(n+2)Z(n+2)$ vanishes. This was expected since all $SU(n+2) \times SU(n+2)$ order parameters must vanish as chiral symmetry is restored.

It is also interesting to study the order parameters defined in the $SU(2) \times SU(2)$ chiral limit $m = 0$. Equations (44) and (41) yield the two-flavour decay constant and quark condensate:

$$Z(2)[1 - \bar{e}_{\pi n}] = \frac{nr}{nr+2}[1 - \eta(r) - e_n] \times \left\{ 1 - y_n k_n(r) g_n + \frac{2}{nr} z_n \right\} \quad (65)$$

$$\xrightarrow{z_n \rightarrow 0} \frac{nr}{nr+2}[1 - \eta(r) - e_n] \times \left\{ 1 - \frac{4k_n(r)g_n}{u_n} \right\}, \quad (66)$$

$$X(2)[1 - \bar{d}_{\pi n}] = \frac{nr}{nr+2}[1 - \epsilon(r) - d_n] \quad (67)$$

$$\times \left\{ 1 - y_n^2 k_n(r) f_n + \frac{2}{nr} y_n z_n \right\} \xrightarrow{z_n \rightarrow 0} \frac{nr}{nr+2}[1 - \epsilon(r) - d_n] \times \left\{ 1 - \frac{16k_n(r)f_n}{u_n^2} \right\}. \quad (68)$$

As long as we are not in the vicinity of the origin $u = v = 0$, we expect $X(2)$ and $Z(2)$ to remain close

to 1 (for $r \geq 20$) along the critical line where the $SU(n+2) \times SU(n+2)$ order parameters vanish. We can understand it by combining the definition of $X(2)$ and $Z(2)$, see (40), with the Ward identities for the masses and decay constants:

$$X(2)[1 - \bar{d}_{\pi n}] = X(n+2) + nr \frac{2m^2 \bar{Z}_n^s}{F_\pi^2 M_\pi^2}, \quad (69)$$

$$Z(2)[1 - \bar{e}_{\pi n}] = Z(n+2) + nr \frac{2m \bar{\xi}_n}{F_\pi^2}. \quad (70)$$

For instance, the two-flavour quark condensate is the sum of its $SU(n+2) \times SU(n+2)$ counterpart and a LEC describing the violation of the OZI rule in the scalar sector. We shall call the first term the ‘‘genuine’’ condensate – stemming directly from the breakdown of $SU(n+2) \times SU(n+2)$ chiral symmetry – and the latter the ‘‘induced’’ condensate – induced by the massive strange-like quark pairs present in the $SU(2) \times SU(2)$ vacuum [24]. The same analysis applies to the decay constants $Z(2)$ and $Z(n+2)$. We see now clearly how $SU(2) \times SU(2)$ chiral symmetry can remain broken while the $SU(n+2)$ critical line is reached. Even though there is no genuine contribution to the two-flavour order parameters, the vacuum is not invariant under $SU(2) \times SU(2)$ chiral rotations because of the symmetry-breaking transitions $\bar{s}_i s_i \rightarrow \bar{u} u + \bar{d} d$ which violate the OZI rule and are suppressed for $N_c \rightarrow \infty$.

Let us now move along the critical line towards the origin $u = v = 0$ by changing the value of the fluctuation parameters u and v (we must be on the second sheet to do so). Equations (66) and (68) indicate a suppression of $X(2)$ and $Z(2)$, leading finally to their vanishing:

$$Z(2) \geq 0 \leftrightarrow u \geq 4kg, \quad X(2) \geq 0 \leftrightarrow u \geq 4\sqrt{kf}. \quad (71)$$

Which order parameter vanishes first depends on the relative size of f and g : if $g \geq (f/k)^{1/2}$, $Z(2)$ vanishes before $X(2)$ does (and the other way round otherwise). The point where at least one of the conditions (71) is fulfilled marks the endpoint of the critical line: if we could proceed further down the critical line, we would end up with an unphysical situation where $X(2)$ or $Z(2)$ is negative.

One can check that the two conditions in (71) are equivalent to the paramagnetic inequalities (62) along the critical line $z = 0$ – where $y = 4/u = 2/(v^{1/2})$ and $SU(n+2) \times SU(n+2)$ order parameters vanish. This was expected, since the paramagnetic inequalities along the critical line yield $X(2) \geq X(n+2) = 0$ and $Z(2) \geq Z(n+2) = 0$, i.e. reduce to positivity constraints for the two-flavour order parameters. On the second sheet, the critical line ends thus at the edge of the domain allowed by the paramagnetic inequalities. In the physical case $n = 1$, and for $r = 25$, the right hand side of Fig. 3 shows that the inequality for v (i.e. for X) is saturated first and that the endpoint of the critical line corresponds to $X(2) = 0$, $Z(2) > 0$. From Appendix B, we see that this occurs in the physical case for any (large) value of r .

In this section, we have investigated the critical line of $SU(n+2) \times SU(n+2)$ symmetry restoration. Along this critical line, both $Z(n+2)$ and $X(n+2)$ vanish, while

$Y(n+2) = X(n+2)/Z(n+2)$ remains non-zero and finite. We have studied the two-flavour order parameters $X(2)$ and $Z(2)$ as well. In the regions of the two sheets allowed by the paramagnetic inequalities, both are different from 0. There is only one exceptional point on the second sheet, where the critical line and the most stringent paramagnetic bound intersect. At this endpoint of the critical line, one of the two $SU(2) \times SU(2)$ order parameters vanishes – $X(2)$ in the physical case $n = 1$.

7 No-fluctuation limit: $n_c \rightarrow \infty$

We briefly sum up in this section the properties of another interesting region in the (u, v) plane where the effect of vacuum fluctuations is suppressed. This can be realised as the large- N_c limit of the theory. Since

$$u = O(1/N_c), \quad v = O(1/N_c), \quad (72)$$

we deal with the vicinity of the origin on the first sheet. The large- N_c limit forces the perturbative solution of the generic system (54) and (55):

$$y = 1 + O(1/N_c), \quad z = 1 + O(1/N_c). \quad (73)$$

The analogy between $1/N_c$ expansion and χ PT ceases here. The large- N_c limit does not have much to say about the expansion of the rescaling factors in (52) and (53) – except perhaps by providing estimates of higher-order counterterms included in the NNLO remainders d_n and e_n . Combining the result (73) with (52) and (53) one gets for large N_c

$$X(n+2) \rightarrow 1 - \epsilon(r) - d_n, \quad Z(n+2) \rightarrow 1 - \eta(r) - e_n, \quad (74)$$

From (41) and (44), we are now able to infer the two-flavour order parameters $X(2)$ and $Z(2)$ in the large- N_c limit. One of course expects to reproduce (74) – any dependence on the number of flavours should disappear at the leading order in $1/N_c$. One first remarks that

$$f_n = O(1/N_c), \quad g_n = O(1/N_c), \quad (75)$$

so that for $N_c \rightarrow \infty$ the right hand sides of (44) and (44) both coincide with the one given in (74). It remains to show that the NNLO remainders $\bar{d}_{\pi n} = \lim_{m \rightarrow 0} d_{\pi n}$ and $\bar{e}_{\pi n} = \lim_{m \rightarrow 0} e_{\pi n}$ are also suppressed for large N_c . Indeed, the large- N_c counting of connected QCD correlation functions yields

$$\lim_{N_c \rightarrow \infty} \frac{\partial}{\partial m_s} \log(F_\pi^2 M_\pi^2) = 0, \quad \lim_{N_c \rightarrow \infty} \frac{\partial}{\partial m_s} \log(F_\pi^2) = 0, \quad (76)$$

order by order in powers of quark masses, since the involved quantities receive contributions only from connected graphs with two (and more) fermion loops. This shows the suppression of $\bar{d}_{\pi n}$ and $\bar{e}_{\pi n}$ and thus the desired result:

$$X(n+2) = X(2) + O(1/N_c), \quad Z(n+2) = Z(2) + O(1/N_c). \quad (77)$$

Both the condensate and the decay constant are independent of the number of massless flavours, and for not too small r both X and Z are close to 1. For completeness, we mention the leading large- N_c behaviour of the LECs $L_5(\mu)$ and $L_8(\mu)$, which can be read off from (31) and (34):

$$L_5 = \frac{F_\pi^2}{8M_\pi^2} [\eta(r) + e'_n] [1 - \eta(r) - e_n] k_n(r), \quad (78)$$

$$L_8 = \frac{F_\pi^2}{16M_\pi^2} [\epsilon(r) + d'_n] [1 - \epsilon(r) - d_n] k_n(r). \quad (79)$$

Notice that the scale dependence only shows up at the next-to-leading order. The OZI-rule suppressed constants L_4 and L_6 remain $O(1)$ at large N_c . They determine how quickly the fluctuation parameters vanish as N_c increases.

8 The limit of large fluctuations

We have seen that both multi-flavour ($N_f \geq 3$) χ PT and $1/N_c$ expansion require small fluctuation parameters u and v . On the other hand, in the real world with three light flavours ($n = 1$), the fluctuation parameters u_1, v_1 , as well as the difference $u_1 - v_1$, are not small compared to 1. Sum-rule studies [11, 12, 14] suggest a significant though small deviation of the LECs $L_6(\mu)$ and $L_4(\mu)$ from their critical values (50) and (51). The large coefficients in (49) then amplify this deviation resulting into fluctuation parameters well above 1. We will now study the effect of large fluctuations on the pattern of chiral symmetry breaking.

8.1 Many-flavour limit

QCD does not contain an obvious parameter which could allow one to describe the limit of large fluctuations. Indeed, in the limit $n \rightarrow \infty$, the effect of sea-quark pairs, such as the induced condensate or the OZI-rule violation in the vacuum channel, are enhanced. However, a large- N_f limit with N_c held fixed would presumably meet phase transitions, leading eventually to the restoration of chiral symmetry and the loss of asymptotic freedom. The definition of a large-fluctuation limit at the level of QCD is thus likely to require a combined limit of large N_f and large N_c . However, at the level of the identities of the effective theory, we are allowed to take the formal limit $n \rightarrow \infty$. We will see that this limit provides a natural example of large fluctuations, if we make the assumption that in this limit the pseudoscalar masses and decay constants remain finite order by order in powers of quark masses m and m_s .

Indeed, with such an assumption, the fluctuation parameters ρ and λ grow with n , unless the constants $L_4^n(\mu)$ and $L_6^n(\mu)$ are set to the asymptotic values of $L_4^{n,\text{crit}}$ and $L_6^{n,\text{crit}}$:

$$L_4^n(\mu) \rightarrow L_4^{\text{as}}(\mu), \quad L_6^n(\mu) \rightarrow L_6^{\text{as}}(\mu), \quad (80)$$

where

$$L_4^{\text{as}}(\mu) = 2L_6^{\text{as}}(\mu) = \frac{1}{256\pi^2} \left(\log \frac{M_K^2}{\mu^2} - \frac{2}{r-1} \log \frac{M_X^2}{M_K^2} \right). \quad (81)$$

From the chiral expansion of the masses and decay constants, (17) and (18) and (25) and (26), and assuming that they remain finite order by order in powers of m and m_s , one can read the a priori unknown large- n behaviour of $Y(n+2)$, Z_n^s , ξ_n , $\tilde{\xi}$ and A_n ,

$$Y(n+2) \sim Z_n^s \sim \tilde{\xi}_n \sim 1/n, \quad A_n \sim \xi_n \sim 1. \quad (82)$$

The constraint on $Y(n+2)$ is imposed by the presence of the logarithmic term (absent in the physical case $n=1$):

$$L'_n \sim -\frac{n}{16\pi^2} \log \frac{M_X^2}{M_K^2} \rightarrow -\infty. \quad (83)$$

This implies in turn that L_6^n should grow like n , whilst L_4^n should approach a finite value, which need not coincide with the critical asymptote (81).

Actually, the same large- n behaviour of the theory is also imposed by the paramagnetic inequalities, provided that f_n and g_n , defined in (42) and (45) are positive:

$$f_n \sim g_n \sim \frac{nr}{32\pi^2} \frac{M_\pi^2}{F_\pi^2} \left[\frac{2}{r-1} \log \frac{M_X^2}{M_K^2} + \log \frac{\bar{M}_K^2}{M_K^2} \right] = nl_\infty \geq 0. \quad (84)$$

The paramagnetic inequality for Z – second inequality in (63) – implies that y vanishes like $1/n$ (or more quickly). The limit $y \rightarrow 0$ is allowed only on the first sheet (y_+) and (60) leads to $v_n - u_n \rightarrow \infty$ like n^2 (or more quickly). On the other hand, the paramagnetic constraint for u – second inequality in (62) – leads to $u = O(n)$, unless a cancellation between kg and E/y occurs. Such a cancellation between the NNLO remainder E and the one-loop chiral logarithms g cannot be logically ruled out. It would however stand against our initial assumption that the chiral expansions exhibit a good overall convergence starting at the NNLO order independently of n .

Let us now describe more precisely the large- n behaviour of u and v . Since u is positive, $z = 1 - uy/4$ cannot diverge. Let us write $z_\infty = \lim_{n \rightarrow \infty} z_n$ and introduce a second parameter a such that

$$u \sim \frac{4n}{a}(1 - z_\infty), \quad v \sim 4n^2/a^2, \quad y \sim a/n. \quad (85)$$

a and z_∞ describe the behaviour of the fluctuation parameters u and v . The order parameters tend to finite values:

$$\begin{aligned} X(n+2) &\rightarrow 0, \\ Z(n+2) &\rightarrow (1 - \eta(r) - e_n)z_\infty, \\ X(2)[1 - \bar{d}_{\pi n}] &\rightarrow 1 - \epsilon(r) - d_n, \\ Z(2)[1 - \bar{e}_{\pi n}] &\rightarrow [1 - \eta(r) - e_n][1 - ak l_\infty]. \end{aligned} \quad (86)$$

In this limit, the LECs associated with the decay constant identities remain scale-dependent – $L_5 = O(n)$ and $L_4 =$

$O(1)$ – whereas the LECs arising in the mass identities become

$$L_6^n(\mu) \sim \frac{n}{16rka^2} \frac{F_\pi^2}{M_\pi^2}, \quad (87)$$

$$L_8^n(\mu) \sim \frac{n^2}{16a^2} \left(\frac{1 - \eta(r) - e_n}{1 - \epsilon(r) - d_n} \right)^2 (\epsilon(r) + d'_n) \frac{F_\pi^2}{M_\pi^2}. \quad (88)$$

A comment is in order here about the double limit of large N_c and N_f , investigated in [24]. If we consider the large- n formulae for the LECs $L_{4,5,6,8}^n$ [see (87) and (88)], we can recover their standard behaviour in the large- N_c limit, provided that $N_c \sim n$ and $a = O(N_c)$, i.e. $a = F_\pi^2/\Lambda^2$ with Λ an N_c -independent scale⁴.

The large- n limit of the theory illustrates therefore how the large-fluctuation limit can be formulated consistently: the multi-flavour quark condensate is then suppressed whereas the two-flavour condensate $X(2)$ remains different from zero (and close to 1 for $r \geq 15$).

8.2 Large fluctuation parameters in three-flavour QCD

The many-flavour limit of the theory discussed above should merely be viewed as a particular realisation of the limit of large fluctuations which is hopefully consistent but not necessarily unique. The limit of large fluctuations could as well be formulated directly in terms of fluctuation and order parameters keeping the number of flavours fixed to $N_f = 3$. In that case ($n=1$), one avoids the presence of extra $n^2 - 1$ Goldstone boson X states arising for $n > 1$. Since the limit is designed in a slightly different way from the previous section, we are ending up with similar but not identical results for the LECs and the two-flavour parameters.

One may infer from the generic (54) and (55) the lines in the (u, v) plane corresponding to a constant value of $z = z_1$ ($0 < z_1 < 1$):

$$u = \frac{4}{y}(1 - z_1), \quad v = \frac{4}{y^2}(1 - yz_1). \quad (89)$$

It is seen that keeping z fixed and setting $y \rightarrow 0$ one reproduces the large-fluctuation behaviour (85). The decay constant $Z(3) = [1 - \eta(r) - e_1]z_1$ remains non-zero, whereas the three-flavour condensate vanishes asymptotically:

$$X(3) = [1 - \eta(r) - e_1]z_1 y \rightarrow 0. \quad (90)$$

In this large-fluctuation limit for $n=1$, all relevant LECs appearing in the original mass and decay constant identities can be predicted in terms of $r = m_s/m$ and of the parameter $Y(3) = X(3)/Z(3) \rightarrow 0$. The leading behaviour of L_5 and L_8 is then

$$L_5 \sim \frac{F_\pi^2}{8M_\pi^2} \frac{\eta(r) + e'_1}{Y(3)}, \quad (91)$$

⁴ Needless to say that the ‘‘holomorphic phase’’ analysed in [24] is outside the scope of the present discussion which is exclusively based on an effective theory describing the breakdown $SU(N_f) \times SU(N_f) \rightarrow SU_V(N_f)$

$$L_8 \sim \frac{F_\pi^2}{16M_\pi^2} \frac{\epsilon(r) + d'_1}{Y^2(3)}. \quad (92)$$

Notice that the OZI-rule violating constant $L_6(\mu)$ is no more suppressed [to be compared with the large- n result (87)]. This is best seen from the ratio L_6/L_8 which in the $y \rightarrow 0$ limit becomes

$$\lim_{y \rightarrow 0} \frac{L_6(\mu)}{L_8(\mu)} = \frac{1 - \epsilon(r) - d_1}{(r+2)[\epsilon(r) + d'_1]}. \quad (93)$$

It amounts to $L_6/L_8 = 0.43, 0.79, 1.53$ for $r = 20, 25, 30$ respectively. The second OZI-rule violating ratio L_4/L_5 still depends on the order parameter $Z(3)$:

$$\lim_{y \rightarrow 0} \frac{L_4(\mu)}{L_5(\mu)} = \frac{1 - \eta(r) - Z(3) - e_1}{(r+2)[\eta(r) + e'_1]}. \quad (94)$$

This ratio could be more suppressed provided $Z(3)$ is close to 1. In the latter case the expression (94) could be rather sensitive to the NNLO remainder e_1 , contrary to the case of the ratio (93) where d_1 competes with 1 (recall that both d'_1 and e'_1 are suppressed by an extra power of $1/r$).

It remains to work out the two-flavour order parameters $X(2)$ and $Z(2)$ in the $y \rightarrow 0$, $n = 1$ large-fluctuation limit:

$$X(2)[1 - \bar{d}_{\pi 1}] = (1 - \epsilon(r) - d_1) \frac{r}{r+2}, \quad (95)$$

$$Z(2)[1 - \bar{e}_{\pi 1}] = (1 - \eta(r) - e_1) \frac{r + 2z_1}{r+2}. \quad (96)$$

Despite the vanishing of the three-flavour condensate $X(3) \rightarrow 0$, the two-flavour condensate $X(2)$ remains non-zero and close to 1, provided $r = m_s/m$ is not too small. This effect is entirely due to the induced condensate and it is proportional to the strange quark mass. It is worth stressing the fundamental difference between the chiral symmetry restoration which occurs along the critical line $v = u^2/4$ for finite u and v and the large-fluctuation limit in which $(u, v) \rightarrow \infty$. This difference merely occurs for three-flavour order parameters: whereas in the symmetry restoration case both $Z(3)$ and $X(3)$ vanish and their ratio $Y(3) = X(3)/Z(3)$ remains non-zero, the large fluctuation limit is characterised by a continuous decrease of $Y(3)$ and of the condensate, with the decay constant $Z(3)$ held fixed. This manifestation of large fluctuations need not correspond to a phase transition: they would lead naturally to a spontaneous breakdown of chiral symmetry with a very small (but non-vanishing) quark condensate. In this context, the recently proposed “no-go” theorems [25] – stating that the vanishing of the condensate implies a vanishing pion decay constant and chiral symmetry restoration – do not necessarily apply.

The limits of small and large fluctuations and of chiral symmetry restoration are summarised in Fig. 4, while the corresponding results for the order parameters are collected in Table 1.

9 Two-flavour chiral perturbation theory

The striking outcome of the previous analysis is the persistence of a large two-flavour condensate $\Sigma(2)$ even when

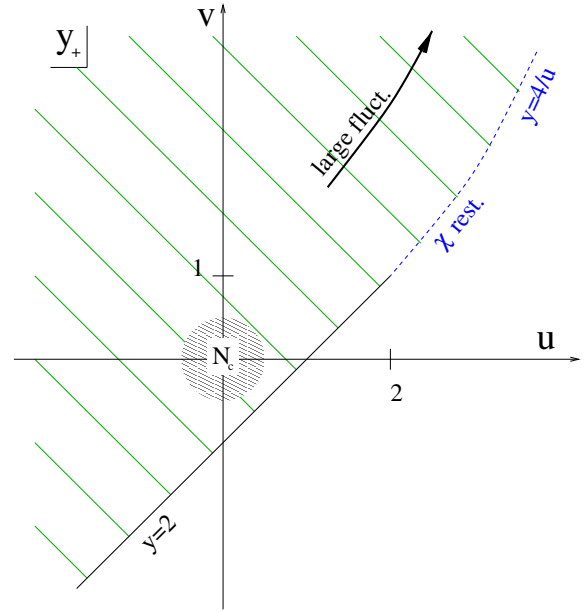


Fig. 4. Limits of small and large fluctuations, and of chiral symmetry restoration on the y_+ sheet for $n = 1$: the shaded region around the origin denoted N_c corresponds to the domain of application for the large- N_c limit, the arrow illustrates the large-fluctuation limit $u, v \rightarrow \infty$, and the “ χ rest.” border is the critical line of chiral symmetry restoration where $z = 0$

the multi-flavour condensate $\Sigma(N_f)$ [$N_f \geq 3$] is suppressed. We have seen that $X(3)$ could be well below 1, indicating that the expansion of $F_\pi^2 M_\pi^2$ in powers of m_u, m_d and m_s need not be dominated by the genuine condensate term $(m_u + m_d)\Sigma(3)$. Actually, the whole $N_f \geq 3$ χ PT treated in the standard way can exhibit instabilities, not because of too large a strange quark mass, but rather because of too sizeable vacuum fluctuations of $q\bar{q}$ pairs. On the other hand, the expansion of the same quantity $F_\pi^2 M_\pi^2$ in powers of m_u, m_d only is expected to be dominated by the $N_f = 2$ condensate term $(m_u + m_d)\Sigma(2)$. Indeed, for not too small $r = m_s/m > 15$, the result $X(2) \sim 1$ is emerging independently of the fluctuation parameters u and v reflecting an important contribution of the induced condensate. This suggests that for suitable pion observables, the standard $SU(2) \times SU(2)$ χ PT [6] could be a well-convergent expansion scheme.

In order to gain more insight into the different behaviour of two- and multi-flavour chiral dynamics, it is convenient to rewrite the $N_f = 2$ Ward identities generating the expansion of $F_\pi^2 M_\pi^2$ and of F_π^2 in a form as close as possible to Sects. 3 and 4. They involve the condensate $\Sigma(2)$, the decay constant $F^2(2)$ and the two $O(p^4)$ symmetry-breaking scale-independent LECs $\bar{\ell}_3$ and $\bar{\ell}_4$ [6]:

$$F_\pi^2 M_\pi^2 = 2m\Sigma(2) + \frac{m^2 B^2(2)}{8\pi^2} (4\bar{\ell}_4 - \bar{\ell}_3) + F_\pi^2 M_\pi^2 \delta, \quad (97)$$

$$F_\pi^2 = F^2(2) + \frac{mB(2)}{4\pi^2} \bar{\ell}_4 + F_\pi^2 \varepsilon. \quad (98)$$

Table 1. Values of chiral order parameters in three different limits ($n = 1$). The second column corresponds to small fluctuations (large- N_c limit). The third one stands for the large-fluctuation limits ($y \rightarrow 0$, z set to z_1). The fourth one indicates chiral symmetry restoration ($z \rightarrow 0$, y set to y_1)

Limit	$N_c \rightarrow \infty$	Large fluct. ($y \rightarrow 0$)	χ rest. ($z \rightarrow 0$)
u_1	0	$4(1 - z_1)/y$	$4/y_1$
v_1	0	$4(1 - yz_1)/y^2$	$4/y_1^2$
$Y(3)$	$\frac{1 - \epsilon(r) - d_1}{1 - \eta(r) - e_1}$	0	$\frac{1 - \epsilon(r) - d_1}{1 - \eta(r) - e_1} y_1$
$Z(3)$	$1 - \eta(r) - e_1$	$z_1[1 - \eta(r) - e_1]$	0
$Z(2)$	$1 - \eta(r) - e_1$	$\frac{r + 2z_1}{r + 2} \frac{1 - \eta(r) - e_1}{1 - \bar{e}_{\pi 1}}$	$\frac{r}{r + 2} \frac{1 - \eta(r) - e_1}{1 - \bar{e}_{\pi 1}} [1 - y_1 k_1(r) g_1]$
$X(3)$	$1 - \epsilon(r) - d_1$	0	0
$X(2)$	$1 - \epsilon(r) - d_1$	$\frac{r}{r + 2} \frac{1 - \epsilon(r) - d_1}{1 - \bar{d}_{\pi 1}}$	$\frac{r}{r + 2} \frac{1 - \epsilon(r) - d_1}{1 - \bar{d}_{\pi 1}} [1 - y_1^2 k_1(r) f_1]$

Here $B(2) = \Sigma(2)/F^2(2)$ and the NNLO remainders δ and ε^5 are $O(m^2)$, expected to be of order 1%. The analogy with the multi-flavour case can be pushed further by rewriting (97) and (98) in the form of (47) and (48):

$$X(2) = 1 - \delta - Y(2)^2 \bar{\rho}/4, \quad Z(2) = 1 - \varepsilon - Y(2) \bar{\lambda}/4, \quad (99)$$

where

$$\bar{\rho} = \frac{1}{8\pi^2} \frac{M_\pi^2}{F_\pi^2} (4\bar{\ell}_4 - \bar{\ell}_3), \quad \bar{\lambda} = \frac{1}{8\pi^2} \frac{M_\pi^2}{F_\pi^2} 4\bar{\ell}_4. \quad (100)$$

The simple rescaling

$$\bar{z} = \frac{1}{1 - \varepsilon} \frac{F^2(2)}{F_\pi^2}, \quad \bar{y} = \frac{1 - \varepsilon}{1 - \delta} \frac{2mB(2)}{M_\pi^2}, \quad (101)$$

$$\bar{u} = \bar{\lambda} \bar{k}, \quad \bar{v} = \bar{\rho} \bar{k}, \quad \bar{k} = \frac{1 - \delta}{(1 - \varepsilon)^2}, \quad (102)$$

brings the two-flavour mass and decay constant identities into the standard form (54) and (55):

$$\bar{z} + \frac{1}{4} \bar{u} \bar{y} = 1, \quad (103)$$

$$\bar{z} + \frac{1}{4} \bar{v} \bar{y} = \frac{1}{\bar{y}}. \quad (104)$$

The formal analogy is now complete. In the two-flavour case the factors $\epsilon(r)$ and $\eta(r)$ have to be omitted in the rescaling factors, but otherwise all equations look identical. Where is the difference?

The multi- and two-flavour cases behave differently because the corresponding parameters v and u (or ρ and λ) are respectively of a different origin and magnitude. We know from the previous discussion that the two-flavour quantity $\bar{y} \sim 1$, and consequently, the parameters $\bar{v} - \bar{u}$

and \bar{u} cannot be too large. This in turn excludes an unlimited growth of $|\bar{\ell}_3|$. As a consequence one may expect the perturbative solution of the fundamental equations (103) and (104) [obtained by Taylor expanding the non-perturbative solution] to make sense.

We have already emphasised the connection of the multi-flavour parameters v_n, u_n ($n \geq 1$) with the correlations between vacuum strange and non-strange $\bar{q}q$ pairs and (last but not least) with the fluctuations of small Euclidean Dirac eigenvalues [4]. We do not know much about the importance of these fluctuations from first principles, but we do understand why in the $N_f \geq 3$ theory such fluctuations manifest themselves through important OZI-rule violations in the vacuum channel $J^P = 0^+$. Hence, if the quark pairs in the vacuum are strongly correlated and/or the low-energy Dirac spectrum is subjected to large fluctuations, the multi-flavour parameters u_n, v_n are likely large and the perturbative solution of the corresponding fundamental equations (54) and (55) breaks down. On the other hand, in a $N_f = 2$ theory and in the presence of massive ($m_s \sim \Lambda_{\text{QCD}}$) strange quarks in the sea, the same cause does not produce the same effect. In this case, the fluctuations of small Dirac eigenvalues are much harder to relate to low-energy observables: the OZI-rule is inoperative in this case, and the scalar correlator $\langle (\bar{u}u)(\bar{d}d) \rangle$ is chirally invariant and not simply related to an observable order parameter. The different nature of the parameters \bar{u} and \bar{v} is further illustrated by a different behaviour in the large- N_c limit: whereas the multi-flavour fluctuation parameters u_n, v_n for $n \geq 1$ are suppressed as $O(1/N_c)$, \bar{u} and \bar{v} behave as $O(1)$ since the constants $\bar{\ell}_3$ and $\bar{\ell}_4$ behave like $O(N_c)$.

The fact that \bar{v}, \bar{u} (or $\bar{\rho}, \bar{\lambda}$) as well as the LECs $\bar{\ell}_3, \bar{\ell}_4$ need not be enhanced by fluctuations of small Dirac eigenvalues can be seen directly by comparing (matching) the multi-flavour mass identity (17) with its $N_f = 2$ counterpart (97). In the former, the vacuum-fluctuation contribution resides in the term Z_n^s containing the LEC $L_6(\mu)$.

⁵ The remainder ε should not be confused with the function $\epsilon(r)$ introduced in Sect. 3.2

(97) is a reexpression of the same mass identity in terms of $SU(2) \times SU(2)$ quantities. Both identities have to match order by order in m . At the first order, one obtains the expression for the two-flavour condensate:

$$\Sigma(2) = \Sigma(n+2) + nm_s \bar{Z}_n^s + \Sigma(2) \bar{d}_{\pi n}, \quad (105)$$

which absorbs the major part (the term proportional to m_s) of the vacuum-fluctuation contribution $(nm_s + 2m)Z^s$. $nm_s \bar{Z}_n^s$ is what we have called the induced condensate [4, 24]. The remaining part of Z^s contributes together with A_n (i.e. L_8) to the $N_f = 2$ LECs $\bar{\ell}_3$ and $\bar{\ell}_4$, but this contribution contains neither m_s nor the flavour factor n . Consequently, there is no particular reason why fluctuations of small Dirac eigenvalues should enhance the parameters \bar{v} and \bar{u} and to destabilise the two-flavour standard χ PT.

The non-perturbative effects in the non-linear system (103) and (104) start to show up for $|\bar{v} - \bar{u}| \sim 1$ corresponding to $|\bar{\ell}_3| \sim 35$. At this point, it is worth stressing that the original standard estimates of $\bar{\ell}_3$, $\bar{\ell}_4$ [6, 7] make use of $N_f = 3$ observables assuming the approximate validity of the OZI-rule in the scalar channel. Since vacuum fluctuations do contribute to $\bar{\ell}_3$, $\bar{\ell}_4$, the standard estimates [6, 7] $\bar{\ell}_3 = 2.9 \pm 2.4$ and $\bar{\ell}_4 = 4.3 \pm 0.9$ could be modified by OZI-rule violating effects.

In order to relate $\bar{\ell}_3$, $\bar{\ell}_4$ to $N_f = 3$ observables including OZI-rule violating vacuum fluctuations, one may proceed as follows. First the two-flavour mass and decay constant identities (97) and (98) are equivalently rewritten to express $\bar{\ell}_3$, $\bar{\ell}_4$ in terms of the order parameters $X(2)$ and $Z(2)$:

$$\bar{\ell}_3 = 32\pi^2 \frac{F_\pi^2 Z(2)}{M_\pi^2 X(2)} \left[1 - \varepsilon - (1 - \delta) \frac{Z(2)}{X(2)} \right], \quad (106)$$

$$\bar{\ell}_4 = 8\pi^2 \frac{F_\pi^2 Z(2)}{M_\pi^2 X(2)} [1 - \varepsilon - Z(2)]. \quad (107)$$

Next, one uses (67) and (65) relating $X(2)$ and $Z(2)$ to the multi-flavour order parameters y_n and z_n . In the case of three flavours ($n = 1$) the latter may be written as

$$Z(2) = \frac{r}{r+2} [1 - \eta(r) - (e_1 - \bar{e}_{\pi 1} \dots)] \times \left\{ 1 - y_1 k_1(r) g_1 + \frac{2}{r} z_1 \right\}, \quad (108)$$

$$X(2) = \frac{r}{r+2} [1 - \epsilon(r) - (d_1 - \bar{d}_{\pi 1} \dots)] \times \left\{ 1 - y_1^2 k_1(r) f_1 + \frac{2}{r} y_1 z_1 \right\}, \quad (109)$$

where the ellipses stand for terms of order $(\bar{e}_{\pi 1})^2$, $(\bar{d}_{\pi 1})^2$ as well as $\bar{e}_{\pi 1} \eta(r)$ and $\bar{d}_{\pi 1} \epsilon(r)$. Since $\bar{e}_{\pi 1}$, $\bar{d}_{\pi 1} = O(m_s^2)$ are expected to be of order 0.1 or less, the dotted terms in (108) and (109) can hardly exceed 1%. Furthermore, the leading $O(m_s^2)$ terms cancel out in the differences $e_1 - \bar{e}_{\pi 1}$ and $d_1 - \bar{d}_{\pi 1}$. As a result, the whole contribution of the NNLO remainders in (108) and (109) is $O(mm_s)$, i.e. they can be expected of order 3%.

It is instructive to exploit the above relations and to estimate the order parameters $X(2)$, $Z(2)$ and the LECs

$\bar{\ell}_3$, $\bar{\ell}_4$ that correspond to the extreme cases of no fluctuation ($N_c \rightarrow \infty$) and large fluctuations ($y_1 \rightarrow 0$). Neglecting all NNLO remainders, one gets for $N_c \rightarrow \infty$

$$r = 20 : \quad X(2) = 0.905, \quad Z(2) = 0.949, \quad (110)$$

$$\bar{\ell}_3 = -7.5, \quad \bar{\ell}_4 = 2.0,$$

$$r = 25 : \quad X(2) = 0.955, \quad Z(2) = 0.959, \quad (111)$$

$$\bar{\ell}_3 = -0.6, \quad \bar{\ell}_4 = 1.5,$$

These estimates are quite compatible with standard expectations [6, 7]. On the other hand, one gets in the limit of large fluctuations (taking $z_1 = 0.7$)

$$r = 20 : \quad X(2) = 0.822, \quad Z(2) = 0.922, \quad (112)$$

$$\bar{\ell}_3 = -20.2, \quad \bar{\ell}_4 = 3.2,$$

$$r = 25 : \quad X(2) = 0.884, \quad Z(2) = 0.938, \quad (113)$$

$$\bar{\ell}_3 = -9.6, \quad \bar{\ell}_4 = 2.4,$$

As expected, large vacuum fluctuations push $\bar{\ell}_3$ towards larger negative values, without however reaching the range in which the two-flavour χ PT would start to diverge. The above estimates should be taken with caution: even the small remainders in (108) and (109) of the order of a few per cent can give an important contribution to $\bar{\ell}_3$ and $\bar{\ell}_4$ since the latter involve $X(2) - Z(2)$ and $1 - Z(2)$ respectively.

The information on low-energy $\pi\pi$ phases extracted from the new E865 Brookhaven K_{e4} experiment [15] combined with older data on $I = 0$ and $I = 2$ S-waves [26], have been used recently to extract the two-flavour order parameters $X(2)$, $Z(2)$ as well as the LECs $\bar{\ell}_3$, $\bar{\ell}_4$ in a model-independent analysis [20] which is merely based on the numerical solution of the Roy equations [16]. The result of this analysis reads

$$X(2) = 0.81 \pm 0.07, \quad Z(2) = 0.89 \pm 0.02, \quad (114)$$

$$\bar{\ell}_3 = -17.8 \pm 15.3, \quad \bar{\ell}_4 = 4.1 \pm 0.9. \quad (115)$$

It is compatible with the estimates given above, and especially with the ones corresponding to the large-fluctuation limit.

We would like to close this section with a comment emphasising the importance of the new precise $\pi\pi$ scattering data recently published [15], forthcoming [27] or expected in the near future [28]. Some time ago it has been pointed out [29] that no experimental test of the actual size of the quark condensate and of the magnitude of the quark mass ratio $r = m_s/m$ was available. No particular mechanism of χ SB was anticipated at that time and a general (less predictive) expansion scheme (generalized chiral perturbation theory or $G\chi$ PT) was proposed to analyse data in a model-independent way. Furthermore, there was no or little experimental evidence for the substantial violation of the OZI rule in the scalar channel that supports today the idea of important vacuum fluctuations and a sizeable difference between the two- and multi-flavour condensates.

On the other hand, the present conclusion that $X(2)$ is close to 1 and its consequences for the standard two-flavour χ PT might appear at first sight as a result that

we could have anticipated using nothing but theoretical arguments similar to those in this paper – contradicting therefore the claims made in [29]. As a matter of fact, this conclusion can be drawn only if one gets extra (and independent) information about the size of the quark mass ratio $r = m_s/m$, excluding small values such as $r \sim 10$. In the present paper, a larger value of r was assumed from the onset. If r happened to be small, the factor $\epsilon(r) = 2(r_2 - r)/(r^2 - 1)$ would get close to 1, affecting the rescaling factors in (52) and (53) and in (56). According to (109), $X(2)$ would then be suppressed independently of the size of vacuum fluctuations. Because of the inequalities $X(2) \geq X(3) \geq 0$, both $X(3)$ and $X(2) - X(3)$ would then be suppressed. Hence, for small r , the vacuum fluctuations and the induced condensate would enhance $X(2)$ less significantly, implying a weaker violation of the OZI rule in the scalar channel. In all cases, a strong correlation between r and $X(2)$ [not $X(3)$] persists [4,13]. Hence an important result of the new and/or forthcoming $\pi\pi$ scattering experiments is – among other issues – to put a lower bound on the quark mass ratio r , ruling out small values of the two-flavour GOR ratio $X(2)$. A precise quantitative statement of this lower bound is subject of a detailed $SU(3) \times SU(3)$ analysis of $\pi\pi$ scattering along the lines of the present paper. Such an analysis is in progress [30].

10 Conclusion

(i) Chiral order parameters may exhibit a strong dependence on the number of light flavours. Such a phenomenon can be interpreted as an important effect of sea-quark pairs, which is related to a large violation of the OZI-rule observed in the scalar channel and reflects significant fluctuations of the lowest modes of the Euclidean Dirac operator. In this paper we have focused on the precise interplay between chiral order and fluctuations.

(ii) We have highlighted the particular role of the strange quark, whose mass $m_s \sim \Lambda_{\text{QCD}}$ is light enough to populate the vacuum with massive $\bar{s}s$ pairs, but heavy enough to have influence on the properties of the $SU(2) \times SU(2)$ chiral limit with $m_u, m_d \rightarrow 0$ and m_s fixed at its physical value. In particular, the two-flavour condensate $\Sigma(2) = -\lim_{m_u, m_d \rightarrow 0} \langle \bar{u}u \rangle$ does receive an extra contribution from the massive vacuum $\bar{s}s$ pairs through the OZI-rule violating and $SU(2) \times SU(2)$ symmetry-breaking correlation $\langle (\bar{u}u)(\bar{s}s) \rangle$. The latter is referred to as the induced condensate. Such contribution is absent in the $SU(3) \times SU(3)$ chiral limit $m_{u,d,s} \rightarrow 0$, since the remaining quarks (c, b, t) are too heavy to contribute significantly to vacuum fluctuations. This may lead to significant differences in the manifestation of chiral symmetry breaking in the two- and three-flavour chiral limits.

(iii) To probe the effect of massive quark pairs on chiral symmetry breaking, we have introduced a theory with two ultralight u, d quarks and n degenerate copies of the (massive) strange quark. QCD can be recovered as the particular case $n = 1$. Because of the mass hierarchy $m_{u,d} \ll m_s \ll \Lambda_H$, two different chiral limits can be considered in this model: the two-flavour limit, where only

$m_{u,d} \rightarrow 0$ but $m_s \neq 0$, and the multi-flavour limit, where all $N_f = n + 2$ light masses vanish.

(iv) The multi-flavour case is characterised by two fluctuation parameters which measure the violation of the OZI-rule in the scalar sector. They are related to the two large- N_c suppressed $O(p^4)$ constants $L_6(\mu)$ and $L_4(\mu)$. We have shown that the Ward identities yield two non-linear relations between the fluctuation parameters and the two basic multi-flavour order parameters: the quark condensate $X(N_f) = 2m\Sigma(N_f)/(F_\pi^2 M_\pi^2)$ and the Goldstone boson decay constant $Z(N_f) = F^2(N_f)/F_\pi^2$ (where $F(N_f)$ is the pion coupling constant F_π^2 in the chiral limit). These relations are a direct consequence of the Ward identities: all higher chiral orders (NNLO and beyond) are absorbed into a finite multiplicative renormalisation (rescaling) of the order and fluctuation parameters. The effect of this renormalisation remains small (i.e. rescaling factors ~ 1) provided that the chiral series globally converge from the NNLO order.

(v) Taking the $SU(2) \times SU(2)$ limit $m_u, m_d \rightarrow 0$ of the mass and decay constant Ward identities, one obtains the two-flavour condensate $X(2)$ and decay constant $Z(2)$ in the presence of n copies of massive s -quark pairs in the sea. We can then compare multi- and two-flavour order parameters as functions of fluctuation parameters:

(1) Multi-flavour chiral symmetry is restored along a critical line in the plane of fluctuation parameters. Along this line, the multi-flavour condensate $X(N_f)$ and decay constant $Z(N_f)$ both vanish, but their ratio X/Z stays finite and non-zero. The two-flavour condensate and pseudoscalar decay constant do not vanish in this limit, except in one exceptional point corresponding to the endpoint of the critical line.

(2) In the case of small fluctuations, we recover a large- N_c , mean-field type behaviour. The order parameters then do not depend on the number of light flavours, and the quark condensate is the dominant signal of chiral symmetry breaking.

(3) In the opposite limit of large fluctuations, the multi-flavour quark condensate $X(N_f)$ tends to zero, but chiral symmetry remains spontaneously broken, since the decay constant $Z(N_f)$ stays away from 0. Correspondingly, the two-flavour order parameters receive large contributions from massive sea-quark loops, so that their size is naturally large, and close to the one expected in the large- N_c limit. We would like to stress that the presence of large fluctuations is not necessarily related to the occurrence of a phase transition: they could constitute a feature of the dynamics of the theory, implying in turn a naturally small size of the quark condensate. In this case the two-flavour condensate would be exclusively made of the induced contribution from the massive quark pairs in the vacuum.

(vi) The usual treatment of $N_f \geq 3$ χ PT considers fluctuation parameters as small, which requires a very precise fine tuning of the $O(p^4)$ constants L_6 and L_4 . On the other hand, the available information about the OZI-rule violation in the scalar channel suggests that fluctuations can be enhanced due to the large effect of sea quarks. Such large fluctuations destabilise chiral expansions by suppressing

the lowest order of the series (three-flavour quark condensate), but they need not spoil the overall convergence. To cope with such a situation, we propose to treat the fluctuation parameters (related to the OZI-rule violating LECs L_4 and L_6) non-perturbatively when expressing the parameters of the effective Lagrangian in terms of observables. In practice it amounts to replacing the perturbative solution of the non-linear relation between order and fluctuation parameters (expressed as an expansion in powers of the latter) by the corresponding exact algebraic solution. In this way one can reexpress the low-energy constants mB_0 , F_0 , L_5 , L_7 and L_8 in terms of the two fluctuation parameters, the quark mass ratio $r = m_s/m$ and higher χ PT contributions arising through the (hopefully convergent) expansion of rescaling factors around 1. This procedure corresponds to a non-perturbative resummation of the fluctuations encoded in L_4 and L_6 .

(vii) We have shown that vacuum fluctuations do not suppress the two-flavour condensate. The $N_f = 2$ GOR ratio $X(2)$ remains close to 1 and the standard two-flavour χ PT is likely a well-convergent expansion scheme, provided that the quark mass ratio r is not too small. A lower bound on r can be inferred from precise low-energy $\pi\pi$ scattering data.

(viii) Additional observables (Goldstone boson scattering, decay form-factors, ...) would have to be analysed along similar lines, considering their chiral expansion up to NNLO and making explicit the non-perturbative dependence on the fluctuation parameters. We will illustrate more precisely this point for $\pi\pi$ and πK scattering parameters in a future publication [30]. Applying this method to a sufficiently large set of precisely measured observables should allow one to pin down the size of vacuum fluctuations, to disentangle the effect of massive $\bar{s}s$ pairs on the pattern of two-flavour chiral symmetry breaking and to determine by how much three-flavour chiral order parameters are suppressed. This will eventually lead to a better understanding of the spontaneous breakdown of chiral symmetry and its dependence on the number and hierarchy of light quarks.

Acknowledgements. We would like to thank N.H. Fuchs for collaboration at an early stage of this work. Various discussions with J. Bijnens, U.-G. Meissner, B. Moussallam and P. Talavera have been highly appreciated. Work partially supported by EEC-TMR contract ERBFMRXCT 98-0169 (EURODAPHNE). SD acknowledges support by PPARC, through grant PPA/G/S/1998/00530. LG acknowledges partial support from European Program HPRN-CT-2000-00149.

Appendix

A Goldstone boson masses in multi-flavour QCD

We want to discuss in this section how to combine the Ward identities for the masses and decay constants of the Goldstone bosons. We are going to see that we need as input M_π , M_K , F_π , F_K and M_η to fix the values of the

$O(p^4)$ LECs as functions of r , B_{0n} and $F^2(n+2)$, and then to obtain the values of the remaining observables F_η^2 , M_X^2 and F_X^2 .

First, we have to spell out all the Ward identities concerning the masses and decay constants of the Goldstone bosons. The masses and decay constant identities for the pion and kaon were given in (17) and (18) and in (25) and (26). The Ward identities for the extra X states can be expressed in the more practical form

$$\frac{F_X^2 M_X^2}{F_K^2 M_K^2} = \frac{4r}{r+1} - r \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} \quad (116)$$

$$- \frac{4r}{r+1} d_{K\pi} + r \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} d_{\pi n} + \frac{F_X^2 M_X^2}{F_K^2 M_K^2} d_{Xn},$$

$$\frac{F_X^2}{F_K^2} = 2 - \frac{F_\pi^2}{F_K^2} + \frac{1}{16\pi^2} \frac{M_\pi^2}{F_K^2} Y(n+2)$$

$$\times \left[\frac{1}{2} \log \frac{M_\eta^2}{M_\pi^2} + \frac{r}{n} \log \frac{M_\eta^2}{M_X^2} \right] + \frac{F_X^2}{F_K^2} e_{Xn}$$

$$+ \frac{F_\pi^2}{F_K^2} e_{\pi n} - 2e_{K\pi}. \quad (117)$$

We have not discussed yet the identities for the η , which can be recast in a form reminiscent of the Gell-Mann–Okubo formula:

$$\begin{aligned} & (n+2)F_\eta^2 M_\eta^2 - 4F_K^2 M_K^2 - (n-2)F_\pi^2 M_\pi^2 \\ & = 4(r-1)m^2 \{ (r-1)(2nZ_n^p + A_n) \\ & + B_{0n}^2 [(r-1)L'_n - L] \} + (n+2)F_\eta^2 M_\eta^2 d_\eta \\ & - 4F_K^2 M_K^2 d_K - (n-2)F_\pi^2 M_\pi^2 d_\pi, \end{aligned} \quad (118)$$

$$\begin{aligned} & (n+2)F_\eta^2 - 4F_K^2 - (n-2)F_\pi^2 \\ & = \frac{2mB_{0n}}{16\pi^2} \left[\left(1 + \frac{2r}{n} \right) \log \frac{M_\eta^2}{M_K^2} - (2n-1) \log \frac{M_K^2}{M_\pi^2} \right. \\ & \left. + 2(n^2-1) \log \frac{M_X^2}{M_K^2} \right] + (n+2)F_\eta^2 e_\eta - 4F_K^2 e_K \\ & - (n-2)F_\pi^2 e_\pi. \end{aligned} \quad (119)$$

The η mass identity involves the new LEC $Z_n^p = 16B_{0n}^2 L_7^n$. We have also introduced the NNLO remainders d_η and e_η .

The Ward identities are therefore expressions of F_P^2 and $F_P^2 M_P^2$ ($P = \pi, K, \eta, X$) in terms of the fundamental parameters $r = m_s/m$, $\Sigma(n+2)$, $F^2(n+2)$, the LECs $L_{4\dots 8}$, chiral logarithms of pseudoscalar masses and NNLO remainders. The pion and kaon identities yield then $L_{4,5,6,8}$ in terms of $r, B_{0n}, F^2(n+2)$, chiral logarithms and NNLO remainders; see (30), (31), (34) and (35). We can obtain a similar expression for L_7 from the identity for $F_\eta^2 M_\eta^2$. From Sect. 4, pion and kaon Ward identities lead to the expression of the parameters B_{0n} and $F^2(n+2)$ as functions of the two fluctuations parameters u, v , the ratio of quark masses r and NNLO remainders [see (54) and (55)].

We can now exploit all the remaining Ward identities to write F_X^2 , M_X^2 and F_η^2 as functions of r, u, v , chiral logarithms of M_P and NNLO remainders. We need thus to

Table 2. Value of M_X/M_K as a function of $Y(4)$ and r for $n = 2$

$Y(4)$	$r = 20$	$r = 25$	$r = 30$
0	1.43	1.37	1.30
0.2	1.44	1.38	1.31
0.5	1.44	1.38	1.31
1	1.46	1.40	1.32
2	1.48	1.42	1.35
4	1.57	1.53	1.46

know the masses $M_{\pi,K,\eta}$, the decay constants $F_{\pi,K}$ and the NNLO remainders. As an illustration of the method, we take the physical values of the masses and decay constants and set the remainders to zero, in order to obtain M_X^2 through an iterative procedure. We start with the approximate value

$$\left. \frac{M_X^2}{M_K^2} \right|_{\text{start}} = \frac{rF_K^2}{2F_K^2 - F_\pi^2} \left[\frac{4}{r+1} - \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} \right]. \quad (120)$$

and iterate (116) and (117) until they converge to M_X . For instance, the values for M_X/M_K at $n = 2$ are collected in Table 2.

B Masses in the $SU(2) \times SU(2)$ chiral limit

To know the masses of the Goldstone bosons in the $SU(2) \times SU(2)$ chiral limit, we take the limit $m \rightarrow 0$ in all the mass and decay constant identities, and reexpress each LEC in this limit in terms of the same LEC with $m \neq 0$ and chiral logarithms of \bar{M}_P^2/M_P^2 ; see (43) and (46). Using the previous relations, \bar{F}_P^2 and \bar{M}_P^2 are functions of (r, u, v) , pseudoscalar masses M_Q^2 and \bar{M}_Q^2 , and remainders. If we keep on setting $M_{\pi,K,\eta}$ and $F_{\pi,K}$ to their physical values, and neglecting NNLO remainders, we can compute all the masses in the $SU(2) \times SU(2)$ limit in terms of the ratio of quark masses r and the fluctuation parameters u and v .

We need the pseudoscalar masses \bar{M}_P^2 in particular to compute the factors f_n and g_n arising in the expression of $SU(2) \times SU(2)$ order parameters (41) and (44). These factors can be written as

$$\begin{aligned} h_n &= \frac{M_\pi^2}{F_\pi^2} \frac{r}{r-1} [L - L'_n], & j &= \frac{M_\pi^2}{F_\pi^2} \frac{r+1}{r-1} \frac{1}{32\pi^2} \log \frac{M_\eta^2}{M_\pi^2}, \\ b_n &= -\frac{M_\pi^2}{F_\pi^2} \frac{nr+2}{2} \Delta Z_n^s, & c_n &= -\frac{M_\pi^2}{F_\pi^2} (nr+2) \Delta \tilde{\xi}_n, \\ f_n &= h_n + b_n, & g_n &= h_n + j + c_n. \end{aligned} \quad (121)$$

Table 3 shows an illustration of the physical case $n = 1$. The quark mass ratio is set to $r = 25$. $Z(3)$ and $Y(3)$ were chosen as parameters, rather than u_1 and v_1 [the two sets are related through (54) and (55)], and we set $Z(3)$ to 1. j is only a function of r . h_n depends on $Y(n+2)$ only through L'_n , which vanishes for $n = 1$. For $r = 25$, we have

Table 3. Mass logarithms defined in (121) and involved in $Z(2)$ and $X(2)$, for $r = 25$, $n = 1$ and $Z(3) = 1$ ($j = 0.021$ and $h_1 = 0.059$). $M_{\pi,K,\eta}$ and $F_{\pi,K}$ are set to their physical values, and all NNLO remainders are neglected

Y	b	c	f	g
0	-0.018	-0.021	0.037	0.062
0.5	-0.013	-0.015	0.043	0.067
1	-0.008	-0.009	0.050	0.072
2	-0.003	-0.002	0.056	0.077
4	-0.009	-0.005	0.053	0.071

Table 4. Combination of chiral logarithms l_∞ defined in (127) and involved in $Z(2)$ and $X(2)$ when $n \rightarrow \infty$. $M_{\pi,K}$ and $F_{\pi,K}$ are set to their physical values, and all NNLO remainders are neglected

a	$r = 20$	$r = 25$	$r = 30$
0	0.003	0.003	0.002
1	0.003	0.003	0.002
2	0.003	0.003	0.002
4	0.004	0.004	0.003
6	0.004	0.004	0.004
10	0.006	0.007	0.007

$j = 0.021$ and $h_1 = 0.059$. Further study shows that f_1 and g_1 are only weakly dependent on Z and r . If we set the masses and decay constants to their physical values and vary n , we observe that f_n and g_n are increasing functions of n .

C Large- n limit

We take now the large- n limit as described in Sect. 8.1. We stress that this limit is considered only at the level of the effective theory, with the additional assumption that the masses and decay constants tend to some particular asymptotic values. For numerical purposes, we will furthermore take these asymptotic values as the physical ones, and set the NNLO remainders to 0. We recall that we have introduced in this limit two parameters z_∞ and a describing the behaviour of $SU(n+2) \times SU(n+2)$ chiral order parameters:

$$\begin{aligned} Z(n+2) &\rightarrow (1 - \eta(r) - e_n) z_\infty, \\ Y(n+2) &\sim \frac{a}{n} \frac{1 - \epsilon(r) - d_n}{n - \eta(r) - e_n}. \end{aligned} \quad (122)$$

The chiral logarithms disappear then from the mass identities (17)–(19). In the decay constant identities (25)–(27), only the logarithms of M_K^2/M_X^2 survive – M_η disappears from the right hand side of all mass and decay constant identities. Due to this simplification, (120) becomes an exact formula for M_X (up to NNLO remainders). For $r = 20, 25, 30$, M_X/M_K is respectively 1.43, 1.37 and 1.30.

As far as the η -meson is concerned, the mass identity (118) and the finiteness of all masses and decay constants at large n result only in a constraint on the large- n behaviour of Z_n^p (i.e. L_n^n). On the other hand, the (LEC-free) relation for the decay constants (119) imposes that $F_\eta = F_\pi$ up to NNLO remainders when $n \rightarrow \infty$. This can be related to the structure of the η -meson: $\lambda_\eta = (1 + 2/n)^{1/2} \cdot \text{diag}[1, 1, -2/n \dots - 2/n]$, so that a similar equality may apply to π and η masses in this limit. Such an assumption is however not necessary for our purposes: as we noticed earlier, the large- n behaviour of M_η does not affect the K and X -meson spectrum since $\log M_\eta^2$ disappears from the Ward identities for the unmixed states.

Following the same lines as in the previous section, we can exploit the Ward identities in the $SU(2) \times SU(2)$ chiral limit to determine the pseudoscalar masses for $m \rightarrow 0$. Let us in particular notice the identities for the kaons:

$$\begin{aligned} \frac{\bar{F}_K^2}{F_\pi^2} &\sim 1 + \left(\frac{r}{2} - 1\right) \eta + \frac{r}{2} e'_n - e_n + \frac{\bar{F}_K^2}{F_\pi^2} \bar{e}_{Kn} \\ &+ \frac{ra}{64\pi^2} \frac{1 - \epsilon(r) - d_n}{1 - \eta(r) - e_n} \frac{M_\pi^2}{F_\pi^2} \end{aligned} \quad (123)$$

$$\begin{aligned} &\times \left[2 \log \frac{M_X^2}{M_X^2} - \frac{2}{r-1} \log \frac{M_X^2}{M_K^2} - \log \frac{\bar{M}_K^2}{M_K^2} \right], \\ \frac{\bar{F}_K^2 \bar{M}_K^2}{F_\pi^2 M_\pi^2} &= \frac{r}{2} \left[1 + \left(\frac{r}{2} - 1\right) \epsilon + \frac{r}{2} d'_n - d_n \right] + \frac{\bar{F}_K^2 \bar{M}_K^2}{F_\pi^2 M_\pi^2} \bar{d}_{Kn}, \end{aligned} \quad (124)$$

and the extra- X states:

$$\begin{aligned} \frac{\bar{F}_X^2}{F_\pi^2} &\sim \frac{F_K^2}{F_\pi^2} \left[2 - \frac{F_\pi^2}{F_K^2} \right] - \frac{ra}{16\pi^2} \frac{1 - \epsilon(r) - d_n}{1 - \eta(r) - e_n} \frac{M_\pi^2}{F_\pi^2} \log \frac{\bar{M}_X^2}{M_X^2} \\ &+ r e'_n - e_n + \frac{\bar{F}_X^2}{F_\pi^2} \bar{e}_{Xn}, \end{aligned} \quad (125)$$

$$\begin{aligned} \frac{\bar{F}_X^2 \bar{M}_X^2}{F_\pi^2 M_\pi^2} &= \frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} \left[\frac{4r}{r+1} - r \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} \right] + r(rd'_n - d_n) \\ &+ \frac{\bar{F}_X^2 \bar{M}_X^2}{F_\pi^2 M_\pi^2} \bar{d}_{Xn}. \end{aligned} \quad (126)$$

Comparing (125) and (126) with (116) and (117) shows that $\bar{F}_X^2 = F_X^2$ and $\bar{M}_X^2 = M_X^2$ in the large- n limit (up to NNLO remainders).

The chiral logarithms involved in the discussion of $X(2)$ and $Z(2)$ reduce now to a single combination of logarithms, independent of the η observables:

$$l_\infty = \lim_{n \rightarrow \infty} \frac{f_n}{n} = \lim_{n \rightarrow \infty} \frac{g_n}{n} \quad (127)$$

$$= \lim_{n \rightarrow \infty} \frac{M_\pi^2}{F_\pi^2} \frac{r}{32\pi^2} \left[\frac{2}{r-1} \log \frac{M_X^2}{M_K^2} + \log \frac{\bar{M}_K^2}{M_K^2} \right]. \quad (128)$$

If we neglect (as a first approximation) the chiral logarithms in (123) – setting $a = 0$ – we obtain

$$l_\infty|_{a=0} = \frac{M_\pi^2}{F_\pi^2} \frac{r}{32\pi^2}$$

$$\begin{aligned} &\times \left\{ \log \frac{r}{2} \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} + \log \frac{1 + (r/2 - 1)\epsilon(r)}{1 + (r/2 - 1)\eta(r)} \right. \\ &+ \frac{2}{r-1} \left[\log \left(4 - \frac{4}{r+1} - r \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} \right) \right. \\ &\left. \left. - \log \left(2 \frac{F_\pi^2}{F_K^2} - 1 \right) \right] \right\}. \end{aligned} \quad (129)$$

In (129), we would naïvely expect the first line to be dominant at large r . This is not actually the case. We have

$$\log \frac{r}{2} \frac{F_\pi^2 M_\pi^2}{F_K^2 M_K^2} = \log \frac{r}{r_2^* + 1}, \quad r_2^* = 2 \frac{M_K^2}{M_\pi^2} - 1 = 24, \quad (130)$$

and $(\epsilon - \eta)(r)$ is precisely changing sign at r_2^* , being positive for $r < r_2^*$ and negative $r > r_2^*$. We see therefore that the two logarithms in the first line of (129) are of opposite signs, exchanging them for $r \sim 24$.

The actual sign of l_∞ is therefore a question of subtle compensation between all the logarithms involved in its expression. Following the procedure outlined above, we have computed the values of l_∞ for various values of r and a , collected in Table 4, with $M_{\pi,K}$ and $F_{\pi,K}$ set to their physical values, and all NNLO remainders neglected. We see that l_∞ remains small and positive in any case.

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